UDK 517.983.5

CONTINUOUS AND GENERALIZED SOLUTIONS OF SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Continuous and generalized solutions of singular equations in Banach spaces are studied. We apply Lyapunov–Schmidt's ideas and the generalized Jordan sets techniques and reduce partial differential-operator equations with the Fredholm operator in the main expression to regular problems. In addition the left and right regularizators of singular operators in Banach spaces and fundamental operators in the theory of generalized solutions of singular equations are constructed.

Keywords: singular PDE, regularizators, distributions, fundamental operator-function.

Dedicated to the anniversary
of 60th birthday
of Professor Georgy A. Sviridyuk

Introduction

In 2002 N. Sidorov and M. Falaleev have described (see [14] chapter 6) applications of Lyapunov–Schmidt's ideas [17] to the theory of ordinary differential operator equations in Banach spaces with the irreversible operator in the main part (briefly, singular DOE). A number of initial-value and boundary-value problems, which model real dynamic processes of filtering, thermal convection, deformation of mechanical systems, electrical engineering (models of Barrenblatt–Zheltova, Kochina, Oskolkov, Hoff, V. Dolexal, M. Korpusov, N. Pletner, A. Svechnikov and others), can be reduced to such equations.

Singular differential operator equations have been investigated in the works by S. Krein, N. Sidorov, B. Loginov, I. Melnikhova, K. Akhmedov, A. Kozhanov, R. Schowalter, G. Sviridyuk, M. Falaleev and others. Extended bibliographies can be found in monographs by N. Sidorov [11], N. Sidorov, B. Loginov, A. Sinitsyn and M. Falaleev [14], R. Cassol and R. Schowalter [1], G. Sviridyuk and V. Fedorov [15].

The problem of applying Lyapunov–Schmidt's ideas to singular differential operator equations having Fredholm operators in the main part had been stated already by L. Lusternik in the course of work of his symposia held at Moscow State University in the mid 1950s and has been solved by N. Sidorov (see [11], chapter 4). It appeared obvious that the analog of the classical branching equation for such equations (see [17]) is a system of differential equations of an infinite order. In view of substantial difficulties, which arise in the process of investigation of this system, the theory of singular DOE is presently far from being completed, moreover, there are few results for the nonlinear case. In the monograph [14] an explication of foundations of the general theory of singular differential operator equations is given. Authors have employed the apparatus of generalized Jordan chains (developed in [17]) and the fundamental operators of singular integro-differential expressions (constructed in [2]), the theory of generalized functions, the Nekrasov–Nazarov's method of undetermined coefficients, which is combined with asymptotic methods of the theory of differential equations with singular points, topological methods and the technique of construction of the regularizator algorithm by N. Sidorov's [11], methods of semigroups and groups with kernels developed by G. Sviridyuk [15]. Such a mixture of diverse methods has given the possibility of investigating a wide class of singular ordinary differential operator equations and classes of partial differential operator equations with the Noether operator in the main part. Some recent general results for singular linear partial differential operator equations have been included to this paper.

Let $x = (t, x')$ be a point in the space R^{m+1}, $x' = (x_1, \dots, x_m)$, $D = (D_t, D_{x_1}, \dots, D_{x_m})$, $\alpha = (\alpha_0, \dots, \alpha_m)$, $\mid \alpha \mid = \alpha_0 + \alpha_1 + \cdots \alpha_m$, where α_i are integer non-negative indices, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial t^{\alpha_0} \dots \partial x_m^{\alpha_m}}$.

We also suppose that $B_\alpha : D_\alpha \subset E_1 \to E_2$ are closed linear operators with dense domains in E_1, $x \in \Omega$, where $\Omega \subset R^{m+1}$, $\mid t \mid \leq T$, $x' \in R^m$, E_1, E_2 are Banach spaces.

It is assumed that $\forall u \in E_1$ the function $B_\alpha(x)u$ is analytical with respect to x' and sufficiently smooth with respect to t.

Consider the following differential operator $L(D) = \sum_{|\alpha| \leq l} B_\alpha(x) D^\alpha$. The operator $\sum_{|\alpha|=l} B_\alpha D^\alpha$ we call the main part of $L(D)$.

We consider the equation

$$L(D)u = f(x), \tag{1}$$

where $f : \Omega \to E_2$ is an analytical function of x' sufficiently smooth with respect to t. The initial value problem for (1), when $E_1 = E_2 = R^n$ and the matrix $B = B_{l0\dots0}$ is not singular, has been thoroughly investigated in fundamental papers by I.G. Petrovsky (see [8]). In the case when the operator B is not invertible the theory of initial and boundary value problems for (1) has not been developed even for the case of finite dimensions. The case with the Fredholm operator B with $dimN(B) \geq 1$ is of special interest. This case, when $x \in R^1$, has been considered from different viewpoints in [11, 7, 15] etc. The case, when $x \in R^{m+1}$, $dimN(B) \geq 1$ has attracted our attention only lately [13]. In general, the standard initial value problem with conditions $D_t^i u|_{t=0} = g_i(x')$, $i = 0, \dots, l-1$ for (1) has no classical solutions for an arbitrary right-hand side $f(x)$.

This does not mean that in the present case we do not have a «correctly» stated problem for eq. (1), which has a unique solution for any right-hand side $f(x)$. For example, the positive result can be obtained by decomposing the space E_1 into a direct sum of subspaces in accordance with the properties of operator coefficients B_α and assigning initial conditions on these subspaces separately. This technique applied in a different situation [16] has been also used in the present work. It is assumed that B is a constant Fredholm operator, and among the coefficients B_α there is a constant operator $A \overset{\text{def}}{\equiv} B_{l_1 0\dots0}$, $l_1 < l$, with respect to which B has a complete $A-$Jordan set.

In Section 1 the sufficient conditions of existence of the unique solution for eq. (1) with the initial conditions

$$D_t^i u|_{t=0} = g_i(x'), \quad i = 0, 1, \dots, l_1 - 1, \tag{2}$$

$$(I - P) D_t^i u|_{t=0} = g_i(x'), \quad i = l_1, \dots, l-1, \tag{3}$$

are obtained, where $g_i(x')$ are analytical functions with values in E_1, $Pg_i(x') = 0$, $i = l_1, \dots, l-1$, and the left and right regularizators of singular operators in Banach spaces are constructed. Here P is the projector of E_1 onto corresponding A-root subspace (see [17] chapter 7). In Section 2 a method of fundamental operators for constructing the generalized solution in the class of Schwarz distributions [9] is considered. These investigations can be useful for the new applications [14, 15, 6] of singular differential systems in mechanics and physics and for the development of the new numerical methods in these areas.

1. Continuous Solutions

The first part of this section gives some auxiliary information from [13], the second part suggests the reduction of eq. (1) to the form of Cauchy–Kovalevskaya, whereas in the third part the theorems of existence and uniqueness of solutions of the problem (1), (2), (3) are proved. In conclusion of the first section, left and right regularizators of singular operators in Banach spaces are constructed.

1.1. Decomposition of Banach spaces, (P, Q)-commutativity of linear operators

Let M_i and N_i be mutually complementary subspaces of Banach spaces E_1 and E_2, i.e. $E_1 = M_1 + N_1,\ E_2 = M_2 + N_2,\ P$ is a projector onto M_1 parallel to N_1, Q is a projector onto M_2 parallel to N_2.

Let A be a linear and, generally speaking, unbounded operator from E_1 in E_2 with the domain of definition dense in E_1.

Definition 1. *Let $A : D \subseteq E_1 \to E_2$. If $PD \subseteq D,\ AM_1 \subseteq M_2,\ A(N_1 \cap D) \subseteq N_2$, then it is said that the operator A is $(P, Q)-$reducible.*

Definition 2. *If each time when $u \in D(A)$, the vector $Pu \in D(A)$ and $APu = QAu$, then they say that the operator A is $(P, Q)-$ commutating.*

The operator A $(P, Q)-$commutating if and only if A are $(P, Q)-$reducible.

Property 1. *Let the operator A be $(P, Q)-$ commutating, and the operator Γ $(Q, P)-$commutating, $R(\Gamma) \subseteq D(A),\ R(A) \subseteq D(\Gamma)$. Hence:*

1. *the operator $A\Gamma$ is $Q-$ commutating, $M_2 \cap D(\Gamma)$ and $N_2 \cap D(\Gamma)$ are its invariant subspaces;*

2. *the operator ΓA is $P-$ commutating, $M_1 \cap D(A)$ and $N_1 \cap D(A)$ are its invariant subspaces.*

Let us further assume that M_1 and M_2 are some finite-dimensional subspaces, $M_1 \subseteq D(A),\ \tilde{P} = \sum_1^n \langle \cdot, \gamma_i \rangle \varphi_i,\ \tilde{Q} = \sum_1^n \langle \cdot, \psi_i \rangle z_i$, furthermore, $\langle \varphi_i, \gamma_k \rangle = \delta_{ik},\ \langle z_i, \psi_k \rangle = \delta_{ik}$, $\{\varphi_i\} \in M_1,\ \{z_i\} \in M_2$. Then the condition of $(\tilde{P}, \tilde{Q})-$commutativity of the operator A implies that $AM_1 \subseteq M_2$. Hence, there exists a matrix $\aleph_A : R^n \to R^n$, such that $A\Phi = \aleph_A Z$, where $\Phi = (\varphi_1, \dots, \varphi_n)',\ Z = (z_1, \dots, z_n)'$. This matrix will be called the matrix of $(\tilde{P}, \tilde{Q})$-commutation of the operator A.

Property 2. *If $A\Phi = \aleph_A Z,\ A^*\Psi = \aleph_B \Upsilon,\ \aleph_A, \aleph_B : R^n \to R^n$, where $\Psi = (\psi_1, \dots, \psi_n)',\ \Upsilon = (\gamma_1, \dots, \gamma_n)'$, then A $(\tilde{P}, \tilde{Q})-$commutates if and only if $\aleph_B = \aleph_A'$.*

Consider now a special case when the basis in M_1 is comprized by the elements $\{\varphi_i^{(j)}\},\ i = \overline{1, n},\ j = \overline{1, p_i}$, which form a complete $A-$Jordan set of the operator B, where B is the Fredholm operator.

Hence $B\varphi_i^{(1)} = 0,\ B\varphi_i^{(j)} = A\varphi_i^{(j-1)},\ i = \overline{1, n},\ j = \overline{2, p_i}$, and there exist $\{\psi_i^{(j)}\}$ such that $B^*\psi_i^{(1)} = 0,\ B^*\psi_i^{(j)} = A^*\psi_i^{(j-1)}$. The system $\{z_i^{(j)}\}$ biorthogonal to $\{\psi_i^{(j)}\}$ will be taken as the basis in $M_2 \subset E_2$.

Let us introduce the projectors

$$P = \sum_{i=1}^{n} \sum_{j=1}^{p_i} \langle \cdot, \gamma_i^{(j)} \rangle \varphi_i^{(j)}, \quad Q = \sum_{i=1}^{n} \sum_{j=1}^{p_i} \langle \cdot, \psi_i^{(j)} \rangle z_i^{(j)}. \tag{4}$$

Property 3 *Let the projectors P and Q be defined by the formulas (4). Hence operators B and A be $(P, Q)-$ commutating, furthermore, the corresponding matrices of $(P, Q)-$ commutation* *are symmetric cell-diagonal ones:* $\aleph_B = diag(B_1, \ldots, B_n)$, $\aleph_A = diag(A_1, \ldots, A_n)$, *where*

$$B_i = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 1 & \ldots & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & \ldots & 1 \\ \ldots & \ldots & \ldots \\ 1 & \ldots & 0 \end{bmatrix}, \quad i = \overline{1, n},$$

if $p_i \geq 2$ *and* $B_i = 0$, $A_i = 1$ *if* $p_i = 1$.

1.2. Reduction of equation (1) to the form of Cauchy–Kovalevskaya

Introduce the denotations $B \stackrel{\text{def}}{=} B_{l0\ldots0}$, $A \stackrel{\text{def}}{=} B_{l_1 0\ldots0}$, where $B_{l0\ldots0}$, $B_{l_1 0\ldots0}$ are constant operators, $l_1 < l$, $D(B) \subseteq D(A)$.

Condition 1 $D(B) \subseteq D(B_\alpha)$ $\forall\alpha$, *the Fredholm operator* B *has a complete* A*-Jordan set* $\varphi_i^{(j)}$, B^* *has a complete* A^**-Jordan set* $\psi_i^{(j)}$, $i = \overline{1, n}$, $j = \overline{1, p_i}$, *and the systems* $\gamma_i^{(j)} \equiv A^*\psi_i^{(p_i+1-j)}, z_i^{(j)} \equiv A\varphi_i^{(p_i+1-j)}$, $i = \overline{1, n}, j = \overline{1, p_i}$, *corresponding to them, are biorthogonal,* $k = p_1 + \ldots p_n$ *is a root number.*

Hence, the formulas (4) define the projectors P and Q respectively onto the root subspaces $E_{1k} = span\{\varphi_i^{(j)}\}$, $E_{2k} = span\{z_i^{(j)}\}$.

Since $E_1 = E_{1k} \oplus E_{1\infty-k}$, any solution of eq. (1) can be represented in the form

$$u(x) = \Gamma v(x) + (C(x), \Phi), \tag{5}$$

where $\Gamma = (B + \sum_{i=1}^n \langle\cdot, \gamma_i^{(1)}\rangle z_i^{(1)})^{-1}$ is a bounded operator from E_2 in E_1,

$$C(x) = (C_1(x), \ldots C_n(x))',\ C_i(x) = (C_{i1}(x), \ldots, C_{ip_i}(x)),$$

$$\Phi = (\Phi_1, \ldots, \Phi_n)',\ \Phi_i = (\varphi_i^{(1)}, \ldots, \varphi_i^{(p_i)}),$$

$$v : \Omega \subset R^{m+1} \to E_{2\infty-k},\ C : \Omega \subset R^{m+1} \to R^k.$$

Since

$$\Gamma z_i^{(j)} = \varphi_i^{(p_i+2-j)},\ \Gamma^*\gamma_i^{(j)} = \psi_i^{(p_i+2-j)},\ j = \overline{1, p_i},$$

$$\varphi_i^{(p_i+1)} \stackrel{\text{def}}{=} \varphi_i^{(1)},\ \psi_i^{(p_i+1)} \stackrel{\text{def}}{=} \psi_i^{(1)},$$

the operator Γ is $(P, Q)-$ commutating.

When subsituting the function (5) into eq. (1), it is possible to obtain the equality

$$D_t^l v + \sum_{|\alpha| \leq l,\ \alpha \neq (l,0,\ldots,0)} B_\alpha(x)\Gamma D^\alpha v + \sum_{|\alpha| \leq l} B_\alpha(x)(D^\alpha C, \Phi) = f(x). \tag{6}$$

Let:

Condition 2 *Each of the coefficients* B_α *satisfy just one of the following three conditions:*

1. B_α *is* $(P, Q)-$*commutating, briefly –* $B_\alpha \in \alpha^0$;
2. $QB_\alpha = 0$, *briefly –* $B_\alpha \in \alpha^1$;
3. $(I - Q)B_\alpha = 0$, *briefly –* $B_\alpha \in \alpha^2$.

Now, by projecting (6) onto $E_{2\infty-k}$, we obtain the equation

$$D_t^l v + \sum_{|\alpha|\le l,\, \alpha\notin\alpha^2,\, \alpha\ne(l,0,\dots,0)} B_\alpha(x)\Gamma D^\alpha v = \tag{7}$$

$$= (I-Q)f(x) - \sum_{|\alpha|\le l,\, \alpha\in\alpha^1} B_\alpha(x)(D^\alpha C, \Phi).$$

By projecting the equation (6) onto E_{2k}, we obtain the system

$$\aleph_{l0\dots0} D_t^l C + \sum_{|\alpha|\le l,\, \alpha\notin\alpha^1,\, \alpha\ne(l,0,\dots,0)} \aleph'_\alpha D^\alpha C = b(x,v). \tag{8}$$

Here the vector function $b:\Omega\to R^k$ is defined by the formula

$$\langle f(x) - \sum_{|\alpha|\le l,\, \alpha\notin\alpha^2} B_\alpha(x)\Gamma D^\alpha v, \Psi\rangle.$$

Therefore, equation (6) is reduced to equation (7) and system (8). This equation (7), as a differential equation with respect to v, has the form of Cauchy-Kovalevskaya.

1.3. Selection of initial conditions. Theorems of existence and uniqueness

Let us find the solution of eq. (1) which would satisfy the initial conditions (2), (3). Since $\Gamma E_{2\infty-k} \subset E_{1\infty-k}$, the solution (5) satisfies the initial conditions (2), (3) if and only if

$$D_t^i v|_{t=0} = \begin{cases} B(I-P)g_i(x'),\ i=0,\dots,l_1-1, \\ Bg_i(x'),\ i=l_1,\dots,l-1, \end{cases} \tag{9}$$

$$D_t^i C|_{t=0} = \beta_i(x'),\ i=0,\dots,l_1-1. \tag{10}$$

Here $\beta_i(x')$ are coefficients of projections $Pg_i(x')$, $i=0,\dots,l_1-1$. Hence, the desired $v(x)$ satisfies the initial-value problem (7), (9) in the Cauchy-Kovalevskaya form, and the desired vector function $C(x)$ satisfies, respectively, the initial-value problem (8), (10).

Consider the following two cases when the initial-value problem (8), (10) also has the Cauchy-Kovalevskaya form.

Case 1. $k=n$.

Hence, in system (8), $\aleph_{l0\dots0}=0$, $\aleph_{l_10\dots0}=E$ is a unique matrix. If $\aleph_\alpha=0$ for $l_1 <|\alpha|\le l$, condition 2 is satisfied for $P=\sum_1^n\langle\cdot,\gamma_i^{(1)}\rangle\varphi_i^{(1)}$, $Q=\sum_1^n\langle\cdot,\psi_i^{(1)}\rangle z_i^{(1)}$,

$$\{\alpha^1=\emptyset\}\vee\{\alpha^2=\emptyset\}\vee\{\max_{\alpha\in(\alpha^1\cup\alpha^2)} |\alpha|< l_1\} \tag{11}$$

then system (8) has the order of l_1 and the Cauchy-Kovalevskaya form.

In this connection, the corresponding initial-value problems (7), (9); (8), (10) have unique solutions.

If $\aleph_\alpha$ are triangular $n\times n-$matrices with zeros on the main diagonal and to the right of it, and condition (11) holds, then system (8) turns out to be a recurrent sequence of equations of the order of l_1 in the Cauchy-Kovalevskaya form.

The reasoning explicated above implies the following

Theorem 1. *Let B be a Fredholm operator, $\langle A\varphi_i^{(1)},\psi_k^{(1)}\rangle=\delta_{ik}$, $i,k=\overline{1,n}$, and let condition 2 for $P=\sum_1^n\langle\cdot,\gamma_i^{(1)}\rangle\varphi_i^{(1)}$, $Q=\sum_1^n\langle\cdot,\psi_i^{(1)}\rangle z_i^{(1)}$ and condition (11) be satisfied. If for $l_1<|\alpha|\le l$ the matrices $\aleph_\alpha$ are either equal to zero or all the matrices have zeros to the right of the main* *diagonal, and for $l_1 <| \alpha |$ these have zeros also on the main diagonal, then problem (1), (2), (3) has a unique solution.*

Case 2. $k > n$.

Now, in the system (8) $\aleph_{l0\ldots0} = \aleph_B$, $\aleph_{l_10\ldots0} = \aleph_A$, where the matrices $\aleph_B$, $\aleph_A$ are as defined above (see section 1.2).

Theorem 2. *Let*

1. *conditions 1, 2 be satisfied, furthermore, in condition 2 $\alpha^1 = \emptyset$ or $\alpha^2 = \emptyset$;*
2. *matrices $\aleph'_\alpha = [\aleph^\alpha_{ik}]^n_{i,k=1}$ are lower block-triangular, i.e. $\aleph^\alpha_{ik} = 0$ for $i < k$;*
3. *there are zeros in each diagonal block $\aleph^\alpha_{ii}$ to the left of the nonmain diagonal, and for $| \alpha |> l_1$ there are zeros also on the nonmain diagonal.*

Then the initial-value problem (1), (2), (3) has a unique solution.

For the purpose of proving it is sufficient to note that under the conditions of Theorem 2 system (8) turns out to be a recurrent sequence of linear differential equations of the order of l_1 in the Cauchy–Kovalevskaya form, and eq. (7) is a differential equation of the order of l_1 in the Cauchy–Kovalevskaya form with the bounded operator coefficients. Note that due to the structure of the matrices $\aleph'_\alpha$ components of the vector function $C : \Omega \to R^k$ are defined in the following sequence $c_{1p_i}, \ldots, c_{11}, c_{2p_2}, \ldots, c_{21}, c_{np_n}, \ldots, c_{n1}$. For a more special situation, details of proving may be found in [12].

1.4. The left and right regularizators of singular operators in Banach spaces

Let A and B be constant linear operators from E_1 to E_2, where E_1 and E_2 are Banach spaces, $x(t)$ is an abstract function, $t \in R_n$ with the values in $E_1(E_2)$. The set of such functions is denoted by $X_t(Y_t)$. Now introduce the operator L_t, defined on X_t and Y_t and which is commutable with operators B, A. The examples of such an operator L_t are differential and integral operators, difference operators and their combinations. Note that if operators are solved with respect to higher order derivatives, then they usually generate correct initial and boundary value problems. In other cases, when operators are unsolved according to higher order derivatives, we encounter singular problems (see subsec. 1.1).

Consider the operator $L_tB - A$, which acts from X_t to Y_t, where B, A are closed linear operators from E_1 to E_2 with dense domains, and $D(B) \subseteq D(A)$. If B is invertable, then the operator L_tB-A can be reduced to regular operator by multiplication on B^{-1}. If B is uninvertible, then $L_tB - A$ is called the singular operator. Let operator B in $L_tB - A$ be Fredholm and $\dim N(B) = n \geq 1$. If $\lambda = 0$ is an isolated singular point of the operator-function $B-\lambda A$, then the operators L_tB-A, BL_t-A admit some regularization. For the purpose of explicit representation of the regularizer we use Schmidt's pseudo resolvent $\Gamma = \hat{B}^{-1}$, where $\hat{B} = B + \sum_{i=1}^n < ., A^*\psi_i^{(p_i)} > A\phi_i^{(p_i)}$. On account of condition 1 (sect. 1) and using the equalities $\phi_i^{(j)} = \Gamma A\phi_i^{(j-1)}$, $\psi_i^{(j)} = \Gamma^* A^*\psi_i^{(j-1)}$, $j = 2, \ldots, p_i$, $i = 1, \ldots, n$ it is easy to verify the following equalities

$$(\Gamma - \sum_{i=1}^{n}\sum_{j=1}^{p_i} L_t^j < ., \psi_i^{(p_i+1-j)} > \phi_i)(L_tB - A) = L_t - \Gamma A,$$

$$(L_tB - A)(\Gamma - \sum_{i=1}^{n}\sum_{j=1}^{p_i} L_t^{p_i+1-j} < ., \psi_i > \phi_i^{(j)}) = L_t - A\Gamma.$$

As a result, we have the following

Theorem 3. *Suppose condition 1 in section 1.2 be satisfied. Then*

$$\Gamma - \sum_{i=1}^{n}\sum_{j=1}^{p_i} L_t^j < ., \psi_i^{(p_i+1-j)} > \phi_i \text{ and } \Gamma - \sum_{i=1}^{n}\sum_{j=1}^{p_i} L_t^{p_i+1-j} < ., \psi_i > \phi_i^{(j)}$$

are the left and right regularizators of $L_tB - A$, respectively.

2. Generalized solutions

In this section we present the main ideas of a new approach to the study of degenerate linear differential equations in Banach spaces. Studies of solvability of the Cauchy problem for these equations in the classes of finitely smooth functions have shown that such problems have smooth (classical) solutions only for certain relations between the input data of the problem, i.e., between initial conditions and right-hand side (of free function) equation. The search for these sufficient conditions, as well as formulas for the solution itself, usually is the goal of such studies. In general case the absence of classic solution naturally leads (in linear case) to the formulation of problems in the class of distributions (generalized functions), since in this case there is no need to match the input data of the problem. Therefore, for linear equations the three problems have been formulated. First we need to allocate classes of generalized functions in Banach spaces in which solutions are unique. Second, we need to develop the technology of the generalized solutions construction. And finally we have to study the relationship between the classic generalized solutions. Such triple problem we study in terms of fundamental operator-functions of degenerate integral-differential operators. In order to find the solutions of differential equations in distributions spaces we employ the fundamental operator function which appears to be the most natural tool.

In order to present the essence of this approach we use the following example of the Cauchy problem for integral-differential equation of the second kind

$$Bu^{(2)}(t) = Au(t) + \int_0^t g(t-s)Au(s)ds + f(t), \tag{12}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{13}$$

where A, B are closed linear operators from E_1 to E_2, with dense domains of definition, $D(B) \subset D(A)$, E_1 and E_2 are Banach spaces, $g(t)$ is continuous function, $f(t)$ is sufficiently smooth function B is Fredholm operator.

Let us introduce the main terminology from [14], which use below.

2.1. Generalized functions in Banach spaces

Let E be Banach space, let E^* be – conjugate Banach space. We call the set of finite infinitly differentiable functions $s(t)$ with values in $K(E^*)$ as the main space $K(E^*)$. The convergence in $K(E^*)$ we introduce as follows. The sequence of functions $s_n(t)$ converge to $s(t)$ in $K(E^*)$ if:

a) $\exists R > 0$ such that $\forall n \in N$ $supps_n(t) \subset [-R, R]$;

b) $\forall \alpha \in N$ for $n \to +\infty$ $\sup\limits_{[-R,R]} \parallel s_n^{(\alpha)}(t) - s^{(\alpha)}(t) \parallel \to 0$.

Generalized function (distribution) with values in Banach space E we call any linear continuous functional defined on $K(E^*)$. The set of all generalized functions with values in E we note as $K'(E)$. Convergence in $K'(E)$ is defined as week (point-wise). Here we follow the classic monograph of V.S.Vladimirov and define the set of generalized functions as $\mathcal{D}'$. The equality of two generalized functions, support of generalized function, multiplication of generalized function on infinitly differentiable function are defined as for classic generalized functions. Any locally Bohner integrable function $f(t)$ with values in E derive the following regular generalized function

$$\Big(f(t), s(t)\Big) = \int_{-\infty}^{+\infty} \langle f(t), s(t)\rangle dt, \ \ \forall s(t) \in K(E).$$

All the generalized functions, which operations can be defined using that rule are called as regular generalized functions. The rest of the generalized functions are called as singular. The classic example of singular generalized function is the Dirac delta-function:

$$\Big(a\delta(t), s(t)\Big) = \langle a, s(0)\rangle dt, \ \ \forall s(t) \in K(E), \ \ \forall a \in E.$$

The distribution set with left-bounded support $(K'_+(E) \subset K'(E))$ we denote as $K'_+(E)$. This class is the most conventional in our studies.

Let E_1, E_2 are the Banach spaces, $A(t) \in \mathcal{C}^\infty$ is operator-function with values in $\mathcal{L}(E_1, E_2)$, $h(t) \in \mathcal{D}'$ is classic generalized function [18]. Then the following multiplication (formal expression) $A(t)h(t)$ is called as generalized operator-function. The following generalized operator-function will correspond to integral-differential operator (12)

$$\mathcal{L}_2(\delta(t)) = B\delta''(t) - A\big(\delta(t) + g(t)\theta(t)\big).$$

Let $f(t) \in K'_+(E_1)$, $h(t) \in \mathcal{D}'_+$, then the generalized function $A(t)h(t) * f(t) \in K'_+(E_2)$ defined as follows

$$\Big(A(t)h(t) * f(t), s(t)\Big) = \Big(h(t), \Big(f(\tau), A^* s(t+\tau)\Big)\Big), \ \ \forall s(t) \in K(E_2)$$

is called as convolution of generalized operator-function $A(t)h(t)$ and generalized function $f(t)$.

This definition is correct since supports of the functions $h(t) \in \mathcal{D}'_+$ и $f(t) \in K'_+(E_1)$ are left bounded. It's proofed using the same scheme as proof of the convolution existence in algebra $\mathcal{D}'_+$ in classical theory of generalized functions [18]. It is to be noted that convolution exists in the distributions space with left bounded support and it has associativity property which we employ to proof the principal statements here.

Let us introduce the key concept. The fundamental operator-function of integral-differential operator $\mathcal{L}_2(\delta(t))$ is called generalized operator-function $\mathcal{E}_2(t)$, which satisfies the following equalities:

$$\mathcal{E}_2(t) * \mathcal{L}_2(\delta(t)) * u(t) = u(t), \quad \forall u(t) \in K'_+(E_1),$$

$$\mathcal{L}_2(\delta(t)) * \mathcal{E}_2(t) * v(t) = v(t), \quad \forall v(t) \in K'_+(E_2).$$

The reason for such construction introduction is as follows. If the fundamental operator-function $\mathcal{E}_2(t)$ is known for integral-differential operator $\mathcal{L}_2(\delta(t))$, then in class $K'_+(E_1)$ exists the unique generalized solution

$$u(t) = \mathcal{E}_2(t) * f(t) \in K'_+(E_1)$$

of

$$\mathcal{L}_2(\delta(t)) * u(t) = f(t), \quad f(t) \in K'_+(E_2).$$

Indeed, if $v(t) \neq u(t)$ is other solution of convolution equation then

$$v(t) = \mathcal{E}_2(t) * \mathcal{L}_2(\delta(t)) * v(t) = \mathcal{E}_2(t) * f(t) = u(t).$$

2.2. Fundamental operator-functions of degenerative integral-differential operators and applications

Theorem 4. *If A, B are closed linear operators from E_1 into $E_2, D(B) \subset D(A), \overline{D(A)} = \overline{D(B)} = E_1$, B is Fredholm operator, $\overline{R(B)} = R(B)$, B has complete A-Jordan set $\{\varphi_i^{(j)},\ i = \overline{1,n},\ j = \overline{1,p_i}\}$ [17], then*

a) 2nd order differential operator $\Big(B\delta''(t) - A\delta(t)\Big)$ on the class $K'_+(E_2)$ has fundamental operator-function

$$\mathcal{E}_1(t) = \Gamma\frac{\sinh(\sqrt{A\Gamma}t)}{\sqrt{A\Gamma}}\left[I - \sum_{i=1}^{n}\sum_{j=1}^{p_i}\langle\cdot,\psi_i^{(j)}\rangle A\varphi_i^{(p_i+1-j)}\right]\theta(t) -$$

$$-\sum_{i=1}^{n}\left[\sum_{k=0}^{p_i-1}\left\{\sum_{j=1}^{p_i-k}\langle\cdot,\psi_i^{(j)}\rangle\varphi_i^{(p_i-k+1-j)}\right\}\delta^{(k)}(t)\right];$$

b) 2nd order integral-differential operator
$\Big(B\delta''(t) - A\big(\delta(t) + g(t)\theta(t)\big)\Big)$ in class $K'_+(E_2)$ has the following fundamental operator-function

$$\mathcal{E}_2(t) = \Gamma\sum_{k=1}^{\infty}\Big(\delta(t) + g(t)\theta(t)\Big)^{k-1} * \frac{t^{2k-1}}{(2k-1)!}\theta(t)(A\Gamma)^{k-1} \times \left[I - \sum_{i=1}^{n}\sum_{j=1}^{p_i}\langle\cdot,\psi_i^{(j)}\rangle A\varphi_i^{(p_i+1-j)}\right] -$$

$$-\sum_{i=1}^{n}\left[\sum_{k=0}^{p_i-1}\left\{\sum_{j=1}^{p_i-k}\langle\cdot,\psi_i^{(j)}\rangle\varphi_i^{(p_i-k+1-j)}\right\}\delta^{(2k)}(t) * \Big(\delta(t) + \mathcal{R}(t)\theta(t)\Big)^{k+1}\right],$$

where $\{\psi_i^{(j)},\ i = \overline{1,n},\ j = \overline{1,p_i}\}$ – A^*-Jordan set of the operator B^*, $\Gamma-$ is the Trenogin-Schmidt [17] operator, $\mathcal{R}(t)$ is resolvent of the kernel $(-g(t)\theta(t))$.

The Cauchy problem (12)-(13) in terms of generalized functions can be presented as following convolution equation

$$\mathcal{L}_2(\delta(t)) * \tilde{u}(t) = f(t)\theta(t) + Bu_1\delta(t) + Bu_0\delta'(t),$$

which is class of distributions with left bounded support $K'_+(E_1)$ has the following unique solution

$$\tilde{u}(t) = \mathcal{E}_2(t) * \Big(f(t)\theta(t) + Bu_1\delta(t) + Bu_0\delta'(t)\Big). \tag{14}$$

Further analysis of the singular and regular components of the expression (14) for generalized solution allows us to obtain the theorems on classic solutions of the problem (12)-(13).

Let us demonstrate that based on the following examples.

Example 1. (Boussinesk–Löve Equation) For equation which model (in 1D case) longitudinal oscillations in thin elastic bar with taking into account the lateral inertia [19],

$$(\lambda - \Delta)v_{tt}(t,\bar{x}) = \alpha^2\Delta v(t,\bar{x}) + f(\bar{x}), \quad \lambda, \alpha \neq 0,$$

where $\bar{x} \in \Omega \subset R^m$, Ω is bounded area with boundary $\partial\Omega$ of the class C^∞, we study the Cauchy-Dirichlet problem in the cylinder $\Omega \times R_+$

$$v\Big|_{t=0} = v_0(\bar{x}), \quad \frac{\partial v}{\partial t}\Big|_{t=0} = v_1(\bar{x}) \quad x \in \Omega$$

$$v\Big|_{\partial\Omega} \equiv 0 \quad (x,t) \in \partial\Omega \times R_+.$$

We can reduce that problem to Cauchy problem (12)-(13) with $g(t) \equiv 0$, if the spaces E_1 and E_2 can be selected as follows

$$E_1 \equiv \overset{\circ}{H}{}^{k+2}[\Omega] \equiv \Big\{u \in W_2^{k+2}: \ u(\bar{x}) = 0, \, \bar{x} \in \partial\Omega\Big\}, \quad E_2 \equiv H^k \equiv W_2^k$$

where $W_p^k \equiv W_p^k(\Omega)$ is Sobolev space $1 < p < \infty$, and let

$$B = \lambda - \Delta, \quad A = \alpha^2\Delta, \quad \lambda \in \sigma(\Delta).$$

Here B is Fedholm operator and lengths of all the A-Jordan chains are 1s, i.e. in the formula for fundamental operator-function $\mathcal{E}_1(t)$ from the theorem $p_i = 1$. Which means that generalized solution (14) do not contains the singular component. The remaining regular component will be classic solution of this problem if the following conditions are fuilfilled

$$(f(\bar{x}) + \alpha^2\lambda v_0(\bar{x}), \varphi_k) = 0, \ \ (v_1(\bar{x}), \varphi_k) = 0 \ \forall\varphi_k: \ \ \lambda = \lambda_k,$$

here φ_k are eigen functions of the Laplace operator, which correspond eigen value $\lambda \in \sigma(\Delta)$.

Example 2. (Equation of viscoelastic plates with memory) Let us address the following equation

$$(\gamma - \Delta)v_{tt}(t,\bar{x}) = -\Delta^2 v(t,\bar{x}) + \int_0^t g(t-s)\Delta^2 v(s,\bar{x})ds + f(t,\bar{x}),$$

where $\bar{x} \in \Omega \subset R^m$, Ω is bounded area with boundary $\partial\Omega$ of the class C^∞, for $m = 2$ и $f(t,\bar{x}) = 0$ such equation describes the oscillation of viscoelastic plates with memory [20]. We follow here the last example and study the Cauchy-Dirichlet problem on cylibder $\Omega \times R_+$

$$v\Big|_{t=0} = v_0(\bar{x}), \quad \frac{\partial v}{\partial t}\Big|_{t=0} = v_1(\bar{x}) \quad x \in \Omega$$

$$v\Big|_{\partial\Omega} \equiv 0 \quad (x,t) \in \partial\Omega \times R_+.$$

Such problem we can reduce to the Cauchy problem (12)–(13), if we select spaces and operators as follows

$$E_1 \equiv \overset{\circ}{H}{}^{k+4}[\Omega] \equiv \Big\{u \in W_2^{k+4}: \ u(\bar{x}) = 0, \, \bar{x} \in \partial\Omega\Big\}, \quad E_2 \equiv H^k \equiv W_2^k$$

$$B = \gamma - \Delta, \quad A = -\Delta^2, \quad \gamma \in \sigma(\Delta).$$

Here (similar with example 1) B is Fredholm operator and lengths of all the A-Jordan chains are equal to 1, i.e. in the formula for fundamental operator-function $\mathcal{E}_2(t)$ from the theorem all $p_i = 1$, i.e. generalized solution (14) does not contain singular component. Hence the remaining component will be the classic solution if the following conditions are fuilfilled

$$\Big(f(0,\bar{x}) - \gamma^2 v_0(\bar{x}), \varphi_k\Big) = 0,$$

$$\left(\frac{\partial f(0,\bar{x})}{\partial t} - \gamma^2 v_1(\bar{x}) + g(0)\gamma^2 v_0(\bar{x}), \varphi_k\right) = 0 \ \forall\varphi_k: \ \ \lambda = \lambda_k,$$

here φ_k are eigen functions of Laplace operator which correspond to eigen value $\lambda \in \sigma(\Delta)$.

3. Conclusion

The approach presented in the paper employs essentially the technique of generalized Jordan sets [16], stable pseudoconverses of Noether operators and $(P, Q)-$commutativity of the operators [13] (in accordance with the Jordan structure of the equation's operator coefficients). This is right the technique that makes it possible to state correct initial-boundary-value problems for the differential equations with partial derivatives and with the Noether (unbounded) operator in the main part, as well as to reduce these problems to regular ones. This approach has given the possibility to construct generalized solutions with the finite singular part and to obtain solutions of a number of classes of singular differential equations in closed form [14, 2]. For the first time such an approach was applied by Sidorov [10] in 1972 for the purpose of constructing the asymptotic of branching solutions of nonlinear singular differential and integro-differential equations. Later the method was developed in a number of works and applied to different problems (see the bibliography in [14]). For the case of matrix coefficients, the technique of pseudoconverses of matrices and differential regularizers was developed in detail in the works by Yu.Ye. Boyarintsev, M.V. Bulatov, V.F. Chistyakov and others on the basis of classical methods of linear algebra, This technique was applied by these authors for the purpose of numerical solving algebro-differential equations. Our method can be applied in a more general situation of unbounded operator coefficients, and so, it can be employed not only for constructing the asymptotic of accurate solutions but also for development of stable numerical methods for some classes of Sobolev-type [15] singular differential equations with partial derivatives for which a theory of numerical methods still does not exist.

The work was supported by Federal Framework Programm ≪Scientific and Teaching Staff of Innovative Russia for 2009-2013≫, State Contract № П696, 20.09.2010.

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НЕПРЕРЫВНЫЕ И ОБОБЩЕННЫЕ РЕШЕНИЯ СИНГУЛЯРНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В БАНАХОВЫХ ПРОСТРАНСТВАХ

Н.А. Сидоров, М.В. Фалалеев

Строятся непрерывные и обобщенные решения сингулярных уравнений в банаховых пространствах. На основе альтернативного метода Ляпунова-Шмидта и обобщенных жордановых наборов дифференциально-операторное уравнение в частных производных с фредгольмовым оператором в главном выражении редуцируется к регулярной задаче. С помощью этой техники построены левые и правые регуляризаторы вырожденных операторов в банаховых пространствах и получены в явном виде фундаментальные операторы ряда классов вырожденных уравнений.

Ключевые слова: сингулярные уравнения, регуляризация, распределения, фундаментальная оператор-функция.

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Received 16 November 2011 year.