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RECENT RESULTS ON THE CAHN – HILLIARD EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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The pure or viscous Cahn – Hilliard equation with possibly singular potentials and dynamic boundary conditions is considered and the well-posedness of the related initial value problem is discussed. Then, a boundary control problem for the viscous Cahn – Hilliard system is studied and first order necessary conditions for optimality are shown. Moreover, the same boundary control problem is addressed for the pure Cahn – Hilliard system, by investigating it and passing to the limit in the analogous results for the viscous Cahn – Hilliard system as the viscosity coefficient tends to zero.

Keywords: Cahn – Hilliard equation; dynamic boundary conditions; phase separation; well-posedness; boundary control problem; optimality conditions.

Dedicated to our friend Angelo Favini
on the occasion of his 70th birthday
with best wishes.

Introduction

The classical Cahn – Hilliard equation and the so-called viscous Cahn – Hilliard equation can be written as

$$\partial_t y - \Delta w = 0 \quad \text{and} \quad w = \tau \partial_t y - \Delta y + \beta(y) + \pi(y) - g \quad \text{in } \Omega \times (0, T), \tag{0.1}$$

according to the case $\tau = 0$ or $\tau > 0$, respectively. Here, $\Omega \subset \mathbb{R}^3$ stands for the bounded smooth domain where the evolution takes place and T denotes some final time.

The set of Cahn – Hilliard equations (0.1) provide a description of the evolution phenomena related to solid-solid phase separations. We refer to, in chronological order, [1–5] for some pioneering contributions on these models and problems. In general, an evolution process goes on diffusively. However, the process of the solid-solid phase separation does not seem to comply with this structure: more precisely, each phase concentrates and the so-called spinodal decomposition occurs. A comparative discussion on the modelling approach for phase separation, spinodal decomposition and mobility of atoms between cells can be found in [6–10]).

About the variables appearing in (0.1), y denotes the *order parameter* and w represents the *chemical potential*. Moreover, β and π are the derivatives of the convex part $\widehat{\beta}$ and of the concave perturbation $\widehat{\pi}$ of a double-well potential $f := \widehat{\beta} + \widehat{\pi}$, and g is a source term. Important examples of f are the everywhere defined regular potential f_{reg} and the logarithmic double-well potential f_{log} given by

$$f_{reg}(r) = \frac{1}{4}(r^2 - 1)^2\,, \quad r \in \mathbb{R}, \tag{0.2}$$

$$f_{log}(r) = (1 + r)\ln(1 + r) + (1 - r)\ln(1 - r) - cr^2\,, \quad r \in (-1, 1), \tag{0.3}$$

where $c > 0$ in (0.3) is large enough in order that f_{log} be nonconvex. Another important example refers to the so-called double-obstacle problem and corresponds to the nonsmooth potential $f_{dobs} : \mathbb{R} \to (-\infty, +\infty]$ specified by

$$f_{dobs}(r) = I_{[-1,1]}(r) - cr^2, \quad r \in \mathbb{R} \tag{0.4}$$

with $c > 0$ and where the indicator function of the interval $[-1, 1]$ fulfills

$$I_{[-1,1]}(r) = 0 \quad \text{if } r \in [-1, 1] \quad \text{and} \quad I_{[-1,1]}(r) = +\infty \quad \text{otherwise.} \tag{0.5}$$

In this case, β is no longer a derivative, but it represents the subdifferential $\partial I_{[-1,1]}$ of the indicator function of the interval $[-1, 1]$, that is,

$$s \in \partial I_{[-1,1]}(r) \quad \text{if and only if} \quad s \begin{cases} \le 0 & \text{if } r = -1, \\ = 0 & \text{if } -1 < r < 1, \\ \ge 0 & \text{if } r = 1. \end{cases} \tag{0.6}$$

We are interested in the coupling of (0.1) with the usual no-flux condition for the chemical potential

$$\partial_n w = 0 \tag{0.7}$$

and with the dynamic boundary condition

$$\partial_n y + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + \beta_\Gamma(y_\Gamma) + \pi_\Gamma(y_\Gamma) = g_\Gamma \tag{0.8}$$

on $\Sigma := \Gamma \times (0, T)$, where

- y_Γ denotes the trace $y_{|\Sigma}$ on the boundary Σ;
- $-\Delta_\Gamma$ stands for the Laplace – Beltrami operator on Γ;
- β_Γ and π_Γ are nonlinearities playing the same role as β and π but now acting on the boundary value of the order parameter;
- finally, g_Γ is a boundary source term with no relation with g acting on the bulk.

We aim to point out that the corresponding initial-boundary value problem

$$\partial_t y - \Delta w = 0 \quad \text{in } Q := \Omega \times (0, T), \tag{0.9}$$

$$w = \tau\, \partial_t y - \Delta y + f'(y) - g \quad \text{in } Q, \tag{0.10}$$

$$\partial_n w = 0 \quad \text{on } \Sigma, \tag{0.11}$$

$$y_\Gamma = y_{|_\Sigma} \quad \text{and} \quad \partial_n y + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + f'_\Gamma(y_\Gamma) = g_\Gamma \quad \text{on } \Sigma, \tag{0.12}$$

$$y(0) = y_0 \quad \text{in } \Omega, \tag{0.13}$$

has been first addressed in [11]. Actually, the Cahn – Hilliard system (0.9) – (0.13), or better some variation of it including dynamic boundary conditions, has drawn much attention in recent years: let us quote [12–16] among other contributions. In particular, the existence and uniqueness of solutions as well as the behavior of the solutions as time goes to infinity have been studied for regular potentials f and $f_\Gamma = \widehat{\beta}_\Gamma + \widehat{\pi}_\Gamma$. Moreover, a wide class of potentials, including especially singular potentials like (0.3) and (0.4), has been considered in [11, 17]: in these two papers the authors were able to overcome the difficulties due to singularities and to show well-posedness results along with the long-time behavior of solutions. The approach of [11, 17] is based on a set of assumptions for β, π and β_Γ, π_Γ that gives the role of the dominating potential to f and entails some technical difficulties.

In this note, we follow a strategy developed in [18] to investigate the Allen – Cahn equation with dynamic boundary conditions, which consists in letting f_Γ be the leading potential with respect to f: it turns out that this approach simplifies the analysis. Moreover, we discuss the optimal boundary control problem for the viscous and pure Cahn – Hilliard equation with dynamic boundary conditions, in analogy with the corresponding contributions for the Allen – Cahn equation (see [19] and [20]). In particular, we review the results proved in the three research papers

- [21] (well-posedness and regularity);
- [22] (optimal control problem for the viscous Cahn – Hilliard equation);
- [23] (optimal control problem for the pure Cahn – Hilliard equation).

The paper [21] contains a number of results on the state system (0.9) – (0.13). More precisely, existence, uniqueness and regularity results are proved in [21] for general potentials that include (0.2) – (0.3), and are valid for both the viscous and pure cases, i.e., by assuming just $\tau \geq 0$. Moreover, if $\tau > 0$, further regularity and properties of the solution are ensured.

On the other hand, the paper [22] deals with a control problem for the state system (0.9) – (0.13) when $\tau > 0$, $g = 0$ and $g_\Gamma = u_\Gamma$, the control being then the source term u_Γ that appears in the dynamic boundary condition (cf. (0.8) and (0.12))

$$\partial_n y + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + \beta_\Gamma(y_\Gamma) + \pi_\Gamma(y_\Gamma) = u_\Gamma \quad \text{on } \Sigma. \tag{0.14}$$

Namely, the cost functional

$$\mathcal{J}(y, y_\Gamma, u_\Gamma) := \frac{b_Q}{2} \|y - z_Q\|^2_{L^2(Q)} + \frac{b_\Sigma}{2} \|y_\Gamma - z_\Sigma\|^2_{L^2(\Sigma)} + \frac{b_0}{2} \|u_\Gamma\|^2_{L^2(\Sigma)} \tag{0.15}$$

is considered, for some given functions z_Q, z_Σ and nonnegative constants b_Q, b_Σ, b_0. The control problem then consists in minimizing $\mathcal{J}(y, y_\Gamma, u_\Gamma)$ subject to the state system and to the constraint $u_\Gamma \in \mathcal{U}_{\text{ad}}$, where the control box $\mathcal{U}_{\text{ad}}$ is specified by

$$\mathcal{U}_{\text{ad}} := \big\{ u_\Gamma \in H^1(0, T; H_\Gamma) \cap L^\infty(\Sigma) :$$

$$u_{\Gamma,\min} \le u_\Gamma \le u_{\Gamma,\max} \text{ a.e. on } \Sigma,\ \|\partial_t u_\Gamma\|_{L^2(\Sigma)} \le M_0 \big\}. \qquad (0.16)$$

Here, the functions $u_{\Gamma,\min}, u_{\Gamma,\max} \in L^\infty(\Sigma)$ and the positive constant M_0 are prescribed in order that the control box $\mathcal{U}_{\rm ad}$ be nonempty: this is guaranteed if, for instance, at least one of $u_{\Gamma,\min}$ or $u_{\Gamma,\max}$ actually belongs to $\mathcal{U}_{\rm ad}$. The existence of an optimal control and first-order necessary conditions for optimality are proved and expressed in terms of the solution of a proper adjoint problem in [22].

These results are then used in [23], where the optimal control problem is discussed for the same state system, but when $\tau = 0$. The technique adopted in [23] essentially consists in starting from the known results for $\tau > 0$ and then letting the parameter τ tend to zero. In doing that, some of the ideas of [20] and [24] are used: indeed, these papers [20,24] deal with the Allen – Cahn and the viscous Cahn – Hilliard equations, respectively, and address similar control problems related to the nondifferentiable double-obstacle potential f_{dobs} defined by (0.4).

Now, we think it is important to recall some related contributions. The paper [25] deals with the well-posedness of the system (0.9) – (0.13) in which also an additional mass constraint on the boundary is imposed. The case of a dynamic boundary condition also of Cahn – Hilliard type, i.e. admitting a chemical potential on the boundary too, has been studied in [26]. Recently, Cahn – Hilliard systems have been rather investigated from the viewpoint of optimal control. In this connection, we refer to [27–29] and point out the contributions [30,31] dealing with the convective Cahn – Hilliard equation; the case with a nonlocal potential is studied in [32]. The paper [33] investigates the second-order optimality conditions for the state system (0.9) – (0.13) when $\tau > 0$, $g = 0$ and $g_\Gamma = u_\Gamma$, starting from the results of [22]. There also exist articles addressing some discretized versions of general Cahn – Hilliard systems, cf. [34,35].

The present paper is organized as follows. In the next section, we list our assumptions, state the problem in a precise form and present our well-posedness and regularity results. In the last section we deal with boundary control problems both for the viscous and the pure case.

1. Well-Posedness and Regularity

In this section, we describe the problem more carefully and present some basic results. As in the Introduction, Ω is the body where the evolution takes place. We assume $\Omega \subset \mathbb{R}^3$ to be open, bounded, connected, and smooth, and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ, ∂_n, ∇_Γ and Δ_Γ stand for the boundary of Ω, the outward normal derivative, the surface gradient and the Laplace – Beltrami operator, respectively. Finally, T is a given finite final time and we use the notation

$$Q := \Omega \times (0,T) \quad \text{and} \quad \Sigma := \Gamma \times (0,T).$$

Now, we specify the assumptions on the structure of our system. In order to include both regular and singular potentials, like the examples (0.2), (0.3) and (0.4) of the Introduction, every potential is split into a convex part and a perturbation, with mild assumptions on the former and regularity assumptions on the latter. So, we assume that

$$\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0,+\infty] \text{ are convex, proper, and l.s.c. and } \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0, \qquad (1.1)$$

$$\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R} \text{ are Lipschitz continuous with } \pi(0) = \pi_\Gamma(0) = 0. \qquad (1.2)$$

We introduce the primitives $\widehat{\pi}$ and $\widehat{\pi}_\Gamma$ of π and π_Γ that vanish at the origin and define the potentials f and f_Γ and the graphs β and β_Γ in $\mathbb{R}\times\mathbb{R}$ as follows

$$\widehat{\pi}(r) := \textstyle\int_0^r \pi(s)\,ds \quad \text{and} \quad \widehat{\pi}_\Gamma(r) := \int_0^r \pi_\Gamma(s)\,ds \quad \text{for } r\in\mathbb{R}, \tag{1.3}$$

$$f := \widehat{\beta} + \widehat{\pi} \quad \text{and} \quad f_\Gamma : \widehat{\beta}_\Gamma + \widehat{\pi}_\Gamma, \tag{1.4}$$

$$\beta := \partial\widehat{\beta} \quad \text{and} \quad \beta_\Gamma := \partial\widehat{\beta}_\Gamma. \tag{1.5}$$

Notice that both β and β_Γ are maximal monotone with some effective domains $D(\beta)$ and $D(\beta_\Gamma)$. Due to (1.1), we have $\beta(0)\ni 0$ and $\beta_\Gamma(0)\ni 0$. Clearly, all the basic examples of the Introduction fit the previuos assumptions. For the graphs β and β_Γ we assume the following compatibility condition

$$D(\beta_\Gamma)\subseteq D(\beta) \quad \text{and} \quad |\beta^\circ(r)| \le \eta|\beta_\Gamma^\circ(r)| + C$$
$$\text{for some } \eta,\ C>0 \text{ and every } r\in D(\beta_\Gamma), \tag{1.6}$$

where $\beta^\circ(r)$ and $\beta_\Gamma^\circ(r)$ are the elements of $\beta(r)$ and $\beta_\Gamma(r)$, respectively, having minimum modulus. Roughly speaking, condition (1.6) is opposite to the one postulated in [11]. On the contrary, it is the same as the one introduced in the paper [18], which however deals with the Allen – Cahn equation.

The above assumptions are sufficient for satisfactory well-posedness results. In order to present them with a simplified notation, we set

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma) \quad \text{and} \quad V_\Gamma := H^1(\Gamma), \tag{1.7}$$

$$\mathcal{V} := \{(v,v_\Gamma)\in V\times V_\Gamma:\ v_\Gamma = v_{|\Gamma}\} \quad \text{and} \quad \mathcal{H} := H\times H_\Gamma, \tag{1.8}$$

and endow these spaces with their natural norms. Furthermore, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between V^*, the dual space of V, and V itself. In the following, it is understood that H is embedded in V^* in the usual way, i.e., such that $\langle u,v\rangle = \int_\Omega uv\,dx$ for every $u\in H$ and $v\in V$.

At this point, we can describe the state problem. For the data, we assume that

$$g\in L^2(0,T;H) \quad \text{and} \quad g_\Gamma\in L^2(0,T;H_\Gamma), \tag{1.9}$$

$$g\in H^1(0,T;H) \quad \text{if } \tau=0, \tag{1.10}$$

$$y_0\in V, \quad y_{0|\Gamma}\in V_\Gamma, \quad \widehat{\beta}(y_0)\in L^1(\Omega) \quad \text{and} \quad \widehat{\beta}_\Gamma(y_{0|\Gamma})\in L^1(\Gamma), \tag{1.11}$$

$$m_0 := (y_0)_\Omega \quad \text{lies in the interior of } D(\beta_\Gamma). \tag{1.12}$$

Our problem consists in looking for a quintuplet $(y,y_\Gamma,w,\xi,\xi_\Gamma)$ such that

$$y\in H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega)) \quad \text{and} \quad \tau\,\partial_t y\in L^2(0,T;H), \tag{1.13}$$

$$y_\Gamma\in H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)\cap L^2(0,T;H^2(\Gamma)), \tag{1.14}$$

$$y_\Gamma(t) = y(t)_{|\Gamma} \quad \text{for a.a. } t\in(0,T), \tag{1.15}$$

$$w\in L^2(0,T;V), \tag{1.16}$$

$$\xi\in L^2(0,T;H) \quad \text{and} \quad \xi\in\beta(y) \quad \text{a.e. in } Q, \tag{1.17}$$

$$\xi_\Gamma\in L^2(0,T;H_\Gamma) \quad \text{and} \quad \xi_\Gamma\in\beta_\Gamma(y_\Gamma) \quad \text{a.e. on } \Sigma, \tag{1.18}$$

and satisfying for a.a. $t\in(0,T)$ the variational equations

$$\langle\partial_t y(t),v\rangle + \int_\Omega \nabla w(t)\cdot\nabla v = 0, \tag{1.19}$$

$$\int_\Omega w(t)v = \int_\Omega \tau\,\partial_t y(t)\,v + \int_\Gamma \partial_t y_\Gamma(t)\,v + \int_\Omega \nabla y(t)\cdot\nabla v + \int_\Gamma \nabla_\Gamma y_\Gamma(t)\cdot\nabla_\Gamma v$$
$$+ \int_\Omega \bigl(\xi(t) + \pi(y(t)) - g(t)\bigr)\,v + \int_\Gamma \bigl(\xi_\Gamma(t) + \pi_\Gamma(y_\Gamma(t)) - g_\Gamma(t)\bigr)\,v \tag{1.20}$$

for every $v \in V$ and every $v \in \mathcal{V}$, respectively, and the Cauchy condition

$$y(0) = y_0\,. \tag{1.21}$$

The light notation $\tau\partial_t y$ stands for $\partial_t(\tau y)$. In particular, it means zero if $\tau = 0$. Clearly, equations (1.19) – (1.20) are the variational formulation of the boundary value problem

$$\partial_t y - \Delta w = 0 \quad \text{and} \quad w \in \tau\,\partial_t y - \Delta y + \beta(y) + \pi(y) - g \quad \text{in } Q, \tag{1.22}$$
$$\partial_n w = 0, \quad y_\Gamma = y_{|\Sigma} \quad \text{and} \quad \partial_n y + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + \beta_\Gamma(y_\Gamma) + \pi_\Gamma(y_\Gamma) \ni g_\Gamma \quad \text{on } \Sigma. \tag{1.23}$$

We notice that the duality pairing that appears in (1.19) can be replaced by a usual integral if $\tau > 0$ thanks to the last (1.13), while it has to be kept as it is in the opposite case due to the low level of regularity of $\partial_t y$.

Remark 1. It is worth to note a fact that is typical for Cahn – Hilliard equations. To this end, if $u \in V^*$ and $\underline{u} \in L^1(0,T;V^*)$, we define their generalized mean values $u^\Omega \in \mathbb{R}$ and $\underline{u}^\Omega \in L^1(0,T)$ by setting

$$u^\Omega := \frac{1}{|\Omega|}\,\langle u, 1\rangle \quad \text{and} \quad \underline{u}^\Omega(t) := \bigl(\underline{u}(t)\bigr)^\Omega \quad \text{for a.a. } t \in (0,T). \tag{1.24}$$

Clearly, the relations in (1.24) give the usual mean values when applied to elements of H or $L^1(0,T;H)$. By testing (1.19) by the constant $1/|\Omega|$, we obtain

$$(\partial_t y(t))_\Omega = 0 \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad y(t)_\Omega = m_0 \quad \text{for every } t \in [0,T] \tag{1.25}$$

with the notations (1.24) and (1.12). Thus, the mean value of y is conserved during the evolution. For that reason, this model has to be included in the class of the so-called conserved models for two phase systems.

Now, we present a number of results proved in [21]. As far as uniqueness and continuous dependence are concerned, we have (see [21, Thm. 2.2]):

Theorem 1. *Assume* (1.1) – (1.5) *and let* $(g_i, g_{\Gamma,i}, y_{0,i})$, $i = 1,2$, *be two sets of data satisfying* (1.9) *and such that* $y_{0,1}, y_{0,2}$ *belong to* V *and have the same mean value. Then, if* $(y_i, y_{\Gamma,i}, w_i, \xi_i, \xi_{\Gamma,i})$ *are any two corresponding solutions to problem* (1.13) – (1.21)*, the inequality*

$$\|y_1 - y_2\|^2_{L^\infty(0,T;V^*)} + \tau\|y_1 - y_2\|^2_{L^\infty(0,T;H)} + \|y_{\Gamma,1} - y_{\Gamma,2}\|^2_{L^\infty(0,T;H_\Gamma)}$$
$$+ \|\nabla(y_1 - y_2)\|^2_{L^2(0,T;H)} + \|\nabla_\Gamma(y_{\Gamma,1} - y_{\Gamma,2})\|^2_{L^2(0,T;H_\Gamma)}$$
$$\le c\Bigl\{\|y_{0,1} - y_{0,2}\|^2_* + \tau\|y_{0,1} - y_{0,2}\|^2_H + \|y_{0,1}{}_{|\Gamma} - y_{0,2}{}_{|\Gamma}\|^2_{H_\Gamma}$$
$$+ \|g_1 - g_2\|^2_{L^2(0,T;H)} + \|g_{\Gamma,1} - g_{\Gamma,2}\|^2_{L^2(0,T;H_\Gamma)}\Bigr\} \tag{1.26}$$

holds true with a constant c that depends only on Ω, T, and the Lipschitz constants of π and π_Γ. In particular, any two solutions to problem (1.13) – (1.21) *have the same components y, y_Γ and ξ_Γ. Moreover, even the components w and ξ of such solutions are the same if β is single-valued.*

The above theorem is proved in [21] and is quite similar to the results stated in [11, Thm. 1 and Rem. 9]. In the latter paper (see [11, Rem. 4 and Rem. 8]), it is also shown that partial uniqueness and conditionally full uniqueness as in the above statement are the best one can prove. As for existence, here is our general result [21, Thm. 2.3].

Theorem 2. *Assume* (1.1) – (1.6) *and* (1.9) – (1.12)*. Then, there exists a quintuplet* $(y, y_\Gamma, w, \xi, \xi_\Gamma)$ *satisfying* (1.13) – (1.18) *and solving problem* (1.19) – (1.21).

Next goal is regularity. First, we want to prove that the components y and y_Γ of the solution to problem (1.19) – (1.21) given by the above theorems also satisfy

$$y \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)) \quad \text{and} \quad \tau\partial_t y \in L^\infty(0,T;H), \tag{1.27}$$
$$y_\Gamma \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)), \tag{1.28}$$

whence also

$$y \in L^\infty(Q) \quad \text{and} \quad y_\Gamma \in L^\infty(\Sigma). \tag{1.29}$$

To this aim, we make further assumptions on the data. Namely

$$g \in H^1(0,T;H) \quad \text{and} \quad g_\Gamma \in H^1(0,T;H_\Gamma), \tag{1.30}$$
$$y_0 \in H^2(\Omega) \quad \text{and} \quad y_{0|_\Gamma} \in H^2(\Gamma), \tag{1.31}$$
$$\text{there exists } \xi_0 \in H \text{ such that } \xi_0 \in \beta(y_0) \text{ a.e. in } Q, \tag{1.32}$$
$$\text{there exists } \xi_{\Gamma,0} \in H_\Gamma \text{ such that } \xi_{\Gamma,0} \in \beta_\Gamma(y_{0|_\Gamma}) \text{ a.e. on } \Sigma, \tag{1.33}$$

and, if $\tau = 0$, we reinforce (1.32) by requiring that

$$\text{the family} \quad \{-\Delta y_0 - \beta_\varepsilon(y_0) - g(0) : \ \varepsilon \in (0, \varepsilon_0)\} \quad \text{is bounded in } V \tag{1.34}$$

for some $\varepsilon_0 > 0$. In (1.34), the symbol β_ε stands for the Yosida regularization of β at level ε (see, e.g., [36, p. 28]). Clearly, in order to ensure (1.34), one can assume that $\Delta y_0 + g(0) \in V$ and that $\beta_\varepsilon(y_0)$ remains bounded in V for ε small enough. A sufficient condition for the latter is the following: there exist $r_\pm, r'_\pm \in \mathbb{R}$ such that $r'_- < r_- \le y_0 \le r_+ < r'_+$ a.e. in Ω, $(r'_-, r'_+) \subset D(\beta)$ and the restriction of β to (r'_-, r'_+) is a single-valued Lipschitz continuous function.

Here is our first regularity result (see [21, Thm. 2.4]). It regards general potentials and both the viscous and pure cases.

Theorem 3. *Assume* (1.1) – (1.6) *on the structure and suppose that the data satisfy* (1.30) – (1.33) *and* (1.12)*. Moreover, assume either $\tau > 0$ or* (1.34)*. Then, there exists a solution to problem* (1.19) – (1.21) *that also satisfies* (1.27) – (1.29) *as well as*

$$w \in L^\infty(0,T;V), \quad \xi \in L^\infty(0,T;H), \quad \xi_\Gamma \in L^\infty(0,T;H_\Gamma)\,. \tag{1.35}$$

The next result regards the viscous case, only, but it still allows general potentials (see [21, Thm. 2.6]).

Theorem 4. *In addition to the assumptions of Theorem 3, suppose that* $\tau > 0$ *and that*

$$g \in L^\infty(Q), \quad g_\Gamma \in L^\infty(\Sigma) \quad \text{and} \quad \beta^\circ(y_0) \in L^\infty(Q). \tag{1.36}$$

Then, there exists a solution to problem (1.19) – (1.21) *that also satisfies* (1.27) – (1.29), (1.35) *and*

$$w \in L^\infty(0,T; H^2(\Omega)) \subset L^\infty(Q) \quad \text{and} \quad \xi \in L^\infty(Q). \tag{1.37}$$

It is worth noting an interesting consequence that holds in the following case:

$$D(\beta) \text{ and } D(\beta_\Gamma) \text{ are the same open interval } I. \tag{1.38}$$

This condition is fulfilled if f and f_Γ are, for instance, the same everywhere defined smooth potential (0.2) or the same logarithmic potential (0.3). On the contrary, potentials whose convex part is an indicator function like (0.4) are excluded. However, (1.38) still allows multi-valued operators β and β_Γ. We observe that, if I is not the whole of $\mathbb{R}$ and r_0 is an end-point of it, then β° has an infinite limit at r_0 since the interval I is open. Hence, the second property in (1.37) yields that $y(x,t)$ remains bounded away from r_0. Moreover, if I is unbounded, one can account for (1.29). As $D(\beta_\Gamma) = D(\beta)$ properties of this type for ξ and y imply similar properties for ξ_Γ. Therefore, if (1.38) holds, the next statement (see [21, Cor. 2.7]) easily follows from the results already presented. Let us recall (1.4)–(1.5) before stating it.

Corollary 1. *In addition to the hypotheses of Theorem 3, assume* $\tau > 0$ *and* (1.38) *on the structure and* (1.36) *on the data. Then, there exists a solution* $(y, y_\Gamma, w, \xi, \xi_\Gamma)$ *to problem* (1.13) – (1.21) *that also satisfies* (1.27) – (1.29), (1.35), (1.37) *and*

$$y(x,t) \in K \quad \text{for a.a. } (x,t) \in Q \text{ and some compact subset } K \subset I,$$
$$\xi_\Gamma \in L^\infty(\Sigma).$$

Moreover, if β *and* β_Γ *are single-valued, the unique solution also satisfies*

$$\beta'(y) \in L^\infty(Q), \quad \beta'_\Gamma(y) \in L^\infty(\Sigma)$$

as well as, if f *and* f_Γ *are* C^2 *functions in addition,*

$$f''(y) \in L^\infty(0,T;V) \quad \text{and} \quad f''_\Gamma(y) \in L^\infty(0,T;V_\Gamma)\,.$$

2. Control Problems

In dealing with control problems, it might be easy to prove the existence of an optimal control, while, in general, it is more difficult to establish first-order necessary conditions for optimality. To this aim, one often needs that the state corresponding to the optimal control under attention is very smooth. For that reason, we reinforce our assumptions on the structure. In particular, we also assume that β and β_Γ satisfy (1.38) and are single-valued smooth function on their common domain. Here are the precise assumptions we add to (1.1) – (1.6):

$$D(\beta) = D(\beta_\Gamma) = (r_-, r_+) \quad \text{with} \quad -\infty \le r_- < 0 < r_+ \le +\infty, \tag{2.1}$$

$$f,\ f_\Gamma \text{ are } C^3 \text{ functions on } (r_-, r_+), \tag{2.2}$$

$$|f'(r)| \le \eta\,|f'_\Gamma(r)| + C \quad \text{for some } \eta,\ C > 0 \text{ and every } r \in (r_-, r_+), \tag{2.3}$$

$$\lim_{r\searrow r_-} f'(r) = \lim_{r\searrow r_-} f'_\Gamma(r) = -\infty \quad \text{and} \quad \lim_{r\nearrow r_+} f'(r) = \lim_{r\nearrow r_+} f'_\Gamma(r) = +\infty. \tag{2.4}$$

Clearly, (2.3) and (2.4) follow from (1.1) – (1.6) if both r_- and r_+ are finite. Notice that, once more, the choices $f = f_{reg}$ and $f = f_{log}$ corresponding to (0.2) and (0.3) are allowed. On the contrary, the double-obstacle potential (0.4) is excluded. It is understood that all the assumptions (1.1) – (1.6) and (2.1) – (2.4) on the structure are in force throughout the whole section.

If the data satisfy (1.30) – (1.33) and (1.12), then the solution is unique and enjoys the following regularity

$$y \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.5}$$

$$\tau\partial_t y \in L^\infty(0,T;H), \tag{2.6}$$

$$y_\Gamma \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)), \tag{2.7}$$

$$r_- < \inf_Q \operatorname{ess} y \le \sup_Q \operatorname{ess} y < r_+, \tag{2.8}$$

$$w \in L^\infty(0,T;H^2(\Omega)). \tag{2.9}$$

In particular, all the components y, y_Γ and w are bounded, as well as $f^{(i)}(y)$ and $f_\Gamma^{(i)}(y_\Gamma)$ for $i = 1,2,3$. We notice that the assumptions on y_0 included in (1.31) and (1.36) mean that

$$y_0 \in H^2(\Omega), \quad y_{0|_\Gamma} \in H^2(\Gamma) \quad \text{and} \quad r_- < y_0(x) < r_+ \quad \text{for every } x \in \overline{\Omega} \tag{2.10}$$

in the present case.

At this point, we can address the corresponding control problem. The state system is (1.13) – (1.21) with $g = 0$ and the control is g_Γ, which we term u_Γ now. We rewrite the full system for clarity:

$$\int_\Omega \partial_t y(t)\, v + \int_\Omega \nabla w(t) \cdot \nabla v = 0, \tag{2.11}$$

$$\begin{aligned} \int_\Omega w(t)\, v = \tau \int_\Omega \partial_t y(t)\, v + \int_\Gamma \partial_t y_\Gamma(t)\, v_\Gamma + \int_\Omega \nabla y(t) \cdot \nabla v + \int_\Gamma \nabla_\Gamma y_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma \\ + \int_\Omega f'(y(t))\, v + \int_\Gamma \bigl(f'_\Gamma(y_\Gamma(t)) - u_\Gamma(t)\bigr)\, v_\Gamma, \end{aligned} \tag{2.12}$$

$$y(0) = y_0, \tag{2.13}$$

where (2.11) and (2.12) hold for a.a. $t \in (0,T)$ and for every $v \in V$ and every $(v, v_\Gamma) \in \mathcal{V}$, respectively. We call (y, y_Γ) the *state* corresponding to the control u_Γ, and this is the most important part of the solution. Indeed, the other components are completely determined by it. The control box $\mathcal{U}_{\rm ad}$ is given by

$$\begin{aligned} \mathcal{U}_{\rm ad} := \ \bigl\{ & u_\Gamma \in H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) : \\ & u_{\Gamma,\min} \le u_\Gamma \le u_{\Gamma,\max} \text{ a.e. on } \Sigma,\ \|\partial_t u_\Gamma\|_2 \le M_0 \bigr\} \end{aligned} \tag{2.14}$$

where the constant M_0 and the functions $u_{\Gamma,\min}$ and $u_{\Gamma,\max}$ satisfy

$$M_0 > 0, \quad u_{\Gamma,\min},\, u_{\Gamma,\max} \in L^\infty(\Sigma) \quad \text{and} \quad \mathcal{U}_{\rm ad} \text{ is nonempty}. \tag{2.15}$$

Finally, given the functions and the constants

$$z_Q \in L^2(Q)\,, \quad z_\Sigma \in L^2(\Sigma) \quad \text{and} \quad b_Q,\, b_\Sigma,\, b_0 \in [0,+\infty), \tag{2.16}$$

we set

$$\mathcal{J}(y, y_\Gamma, u_\Gamma) := \frac{b_Q}{2}\,\|y - z_Q\|^2_{L^2(Q)} + \frac{b_\Sigma}{2}\,\|y_\Gamma - z_\Sigma\|^2_{L^2(\Sigma)} + \frac{b_0}{2}\,\|u_\Gamma\|^2_{L^2(\Sigma)} \tag{2.17}$$

for, say, $y \in C^0([0,T];H)$, $y_\Gamma \in C^0([0,T];H_\Gamma)$ and $u_\Gamma \in L^2(\Sigma)$. At this point, the control problem consists in minimizing the cost functional (2.17) subject to the constraint $u_\Gamma \in \mathcal{U}_{\mathrm{ad}}$ and to the state system (2.11) – (2.13). The following result holds true (see [22, Thm. 2.3] for the viscous case and [23, Thm. 2.5] for the pure one):

Theorem 5. *Assume* (2.10)*. Then, there exists* $\overline{u}_\Gamma \in \mathcal{U}_{\mathrm{ad}}$ *such that*

$$\mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma) \le \mathcal{J}(y, y_\Gamma, u_\Gamma) \quad \textit{for every } u_\Gamma \in \mathcal{U}_{\mathrm{ad}}, \tag{2.18}$$

where $\overline{y}$*,* $\overline{y}_\Gamma$*,* y *and* y_Γ *are the components of the solutions* $(\overline{y}, \overline{y}_\Gamma, \overline{w})$ *and* (y, y_Γ, w) *to the state system* (1.13) – (1.21) *corresponding to the controls* $\overline{u}_\Gamma$ *and* u_Γ*, respectively.*

Once such an existence result is established, one looks for necessary conditions for a given $\overline{u}_\Gamma$ to be an optimum control. The natural strategy is the introduction of suitable Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with the following properties: *i*) the control box $\mathcal{U}_{\mathrm{ad}}$ is a closed subset of $\mathcal{X}$; *ii*) for every u_Γ in some neighbourhood $\mathcal{U}$ of $\mathcal{U}_{\mathrm{ad}}$, the state system has a unique solution and the corresponding pair (y, y_Γ) belongs to $\mathcal{Y}$; *iii*) the map $\mathcal{S}$ that associates such a pair (y, y_Γ) to the arbitrary $u_\Gamma \in \mathcal{U}$ is Fréchet differentiable.

This project is difficult to realize in the general case, due to the low regularity of the time derivative of the state, which only belongs to $L^2(0,T;V^*)$ (see (2.5)). The situation is different in the viscous case due to (2.6).

So, we split our discussion in two parts, and we first assume that $\tau > 0$. Then, the results corresponding to the above program are proved in [22] with the following choice of the spaces:

$$\mathcal{X} := H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) \quad \text{and} \quad \mathcal{Y} := H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}). \tag{2.19}$$

Moreover, $\mathcal{U}$ is an arbitrary open neighbourhood of $\mathcal{U}_{\mathrm{ad}}$ (see [22, Prop. 2.4 and Thm. 4.2]). Then, since the functional to be minimized is $\mathcal{U}_{\mathrm{ad}} \ni u_\Gamma \mapsto \widetilde{\mathcal{J}}(u_\Gamma) := \mathcal{J}(\mathcal{S}(u_\Gamma), u_\Gamma)$ and $\mathcal{U}_{\mathrm{ad}}$ is convex, the natural necessary condition is the following: $\langle D\widetilde{\mathcal{J}}(\overline{u}_\Gamma), v_\Gamma - \overline{u}_\Gamma \rangle \ge 0$ for every $v_\Gamma \in \mathcal{U}_{\mathrm{ad}}$, where $D\widetilde{\mathcal{J}}(\overline{u}_\Gamma) \in \mathcal{X}^*$ is the Fréchet derivative of $\widetilde{\mathcal{J}}$ at $\overline{u}_\Gamma$. However, because of the chain rule, this contains the value at $h_\Gamma := v_\Gamma - u_\Gamma$ of the Fréchet derivative $D\mathcal{S}(\overline{u}_\Gamma)$, which turns out to be the solution to the problem obtained by linearizing (1.13) – (1.21) around $\overline{u}_\Gamma$ and taking h_Γ in the linear term that corresponds to the position of the control in the nonlinear problem (see [22, Prop. 6.1 and formula (2.42)]). This can be eliminated by introducing a proper adjoint problem. We set for brevity

$$\varphi_Q = b_Q(\overline{y} - z_Q) \quad \text{and} \quad \varphi_\Sigma = b_\Sigma(\overline{y}_\Gamma - z_\Sigma), \tag{2.20}$$

where $(\overline{y}, \overline{y}_\Gamma)$ is the state associated to the optimal control $\overline{u}_\Gamma$ under attention. Then, the adjoint problem is the following: find a triplet (p, q, q_Γ) that fulfills the regularity requirements

$$p \in H^1(0,T;H^2(\Omega)) \cap L^2(0,T;H^4(\Omega)), \tag{2.21}$$

$$q \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \tag{2.22}$$
$$q_\Gamma \in H^1(0,T;H_\Gamma) \cap L^2(0,T;H^2(\Gamma)), \tag{2.23}$$
$$q_\Gamma(t) = q(t)_{|\Gamma} \quad \text{for a.a. } t \in (0,T), \tag{2.24}$$

and solves the variational equations

$$\int_\Omega q\, v = \int_\Omega \nabla p \cdot \nabla v \quad \text{a.e. in } (0,T) \text{ and for all } v \in V, \tag{2.25}$$

$$\begin{aligned} &- \int_\Omega \partial_t \bigl(p + \tau q\bigr)\, v + \int_\Omega \nabla q \cdot \nabla v + \int_\Omega f''(\overline{y})\, q\, v \\ &\quad - \int_\Gamma \partial_t q_\Gamma\, v_\Gamma + \int_\Gamma \nabla_\Gamma q_\Gamma \cdot \nabla_\Gamma v_\Gamma + \int_\Gamma f''_\Gamma(\overline{y}_\Gamma)\, q_\Gamma\, v_\Gamma = \int_\Omega \varphi_Q\, v + \int_\Gamma \varphi_\Sigma\, v_\Gamma \\ &\quad \text{a.e. in } (0,T) \text{ and every } (v, v_\Gamma) \in \mathcal{V} \end{aligned} \tag{2.26}$$

and the final condition

$$\int_\Omega (p + \tau q)(T)\, v + \int_\Gamma q_\Gamma(T)\, v_\Gamma = 0 \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{2.27}$$

We have the following result (see [22, Thm. 2.5]):

Theorem 6. *Assume* (2.10) *and* $\tau > 0$, *and let* $\overline{u}_\Gamma$ *and* $(\overline{y}, \overline{y}_\Gamma) = \mathcal{S}(\overline{u}_\Gamma)$ *be an optimal control and the corresponding state. Then the adjoint problem* (2.25) – (2.27) *has a unique solution* $(p^\tau, q^\tau, q^\tau_\Gamma)$ *satisfying the regularity conditions* (2.21) – (2.24).

Finally, the necessary condition involving the linearized problem takes a particularly simple form if the solution of the adjoint problem is used. Namely, we have (see [22, Thm. 2.6])

Theorem 7. *Assume* (2.10) *and* $\tau > 0$, *and let* $\overline{u}_\Gamma$ *be an optimal control. Moreover, let* $(\overline{y}, \overline{y}_\Gamma) = \mathcal{S}(\overline{u}_\Gamma)$ *and* $(p^\tau, q^\tau, q^\tau_\Gamma)$ *be the associate state and the unique solution to the adjoint problem* (2.25) – (2.27) *given by Theorem 6. Then we have*

$$\int_\Sigma (q^\tau_\Gamma + b_0 \overline{u}_\Gamma)(v_\Gamma - \overline{u}_\Gamma) \geq 0 \quad \textit{for every } v_\Gamma \in \mathcal{U}_{\mathrm{ad}}. \tag{2.28}$$

Remark 2. In particular, if $b_0 > 0$, (2.28) says that

$$\overline{u}_\Gamma \ \textit{ is the orthogonal projection of } -q^\tau_\Gamma / b_0 \ \textit{ on } \ \mathcal{U}_{\mathrm{ad}} \tag{2.29}$$

with respect to the standard scalar product in $L^2(\Sigma)$.

The next step is to treat the pure Cahn – Hilliard system, i.e., the case $\tau = 0$, and this is done in [23]. The idea is to take the limit as $\tau \searrow 0$ in the above results.

Even though the adjoint problem (2.25) – (2.27) involves a triplet $(p^\tau, q^\tau, q^\tau_\Gamma)$ as an adjoint state, only the third component q^τ_Γ enters the necessary condition (2.28) for optimality. On the other hand, q^τ and q^τ_Γ are strictly related to each other. Hence, we mention the result proved in [22] that deals with the pair (q^τ, q^τ_Γ). To this end, we recall a tool, the generalized Neumann problem solver $\mathcal{N}$, that is often used in connection with the Cahn – Hilliard equations. With the notation for the mean value introduced in (1.24), we define

$$\operatorname{dom}\mathcal{N} := \{v_* \in V^* :\ v_*^\Omega = 0\} \quad \text{and} \quad \mathcal{N} : \operatorname{dom}\mathcal{N} \to \{v \in V :\ v^\Omega = 0\} \tag{2.30}$$

by setting, for $v_* \in \operatorname{dom}\mathcal{N}$,

$$\mathcal{N}v_* \in V, \quad (\mathcal{N}v_*)^\Omega = 0, \quad \text{and} \quad \int_\Omega \nabla\mathcal{N}v_* \cdot \nabla z = \langle v_*, z\rangle \quad \text{for every } z \in V. \tag{2.31}$$

Thus, $\mathcal{N}v_*$ is the solution v to the generalized Neumann problem for $-\Delta$ with datum v_* that satisfies $v^\Omega = 0$. Indeed, if $v_* \in H$, the above variational equation means that $-\Delta\mathcal{N}v_* = v_*$ and $\partial_n\mathcal{N}v_* = 0$. As Ω is bounded, smooth, and connected, it turns out that (2.31) yields a well-defined isomorphism. Furthermore, we introduce the spaces $\mathcal{H}_\Omega$ and $\mathcal{V}_\Omega$ by setting

$$\mathcal{H}_\Omega := \{(v, v_\Gamma) \in \mathcal{H} :\ v^\Omega = 0\} \quad \text{and} \quad \mathcal{V}_\Omega := \mathcal{H}_\Omega \cap \mathcal{V}, \tag{2.32}$$

and endow them with their natural topologies as subspaces of $\mathcal{H}$ and $\mathcal{V}$, respectively. We have the following result.

Theorem 8. *Assume $\tau > 0$. Then, with the notation* (2.20)*, there exists a unique pair (q^τ, q^τ_Γ) satisfying the regularity conditions*

$$q^\tau \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \quad \textit{and} \quad q^\tau_\Gamma \in H^1(0,T;H_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \tag{2.33}$$

and solving the following problem:

$$(q^\tau, q^\tau_\Gamma)(t) \in \mathcal{V}_\Omega \quad \textit{for every } t \in [0,T], \tag{2.34}$$

$$\begin{aligned} &- \int_\Omega \partial_t\big(\mathcal{N}(q^\tau) + \tau q^\tau\big)\, v + \int_\Omega \nabla q^\tau \cdot \nabla v + \int_\Omega f''(\overline{y}^\tau)\, q^\tau\, v \\ &\qquad - \int_\Gamma \partial_t q^\tau_\Gamma\, v_\Gamma + \int_\Gamma \nabla_\Gamma q^\tau_\Gamma \cdot \nabla_\Gamma v_\Gamma + \int_\Gamma f''_\Gamma(\overline{y}^\tau_\Gamma)\, q^\tau_\Gamma\, v_\Gamma \\ &= \int_\Omega \varphi_Q v + \int_\Gamma \varphi_\Sigma v_\Gamma \quad \textit{a.e. in } (0,T) \textit{ and for every } (v, v_\Gamma) \in \mathcal{V}_\Omega, \end{aligned} \tag{2.35}$$

$$\int_\Omega \big(\mathcal{N}q^\tau + \tau q^\tau\big)(T)\, v + \int_\Gamma q_\Gamma(T)\, v_\Gamma = 0 \quad \textit{for every } (v, v_\Gamma) \in \mathcal{V}_\Omega. \tag{2.36}$$

Moreover, the pair (q^τ, q^τ_Γ) is the same as the couple of components of the unique solution $(p^\tau, q^\tau, q^\tau_\Gamma)$ to the adjoint problem (2.25) – (2.27) *given by Theorem 6.*

Remark 3. It is worth to notice that our presentation does not follow [22] in the detail. Indeed, [22] uses this problem to solve the adjoint problem (2.25) – (2.27) as follows. From one hand, the system (2.34) – (2.36) can be seen as a backward Cauchy problem in the framework of the Hilbert triplet $(\mathcal{V}_\Omega, \mathcal{H}_\Omega, \mathcal{V}^*_\Omega)$ (see [22, formula (5.25)]). Thus, one proves that it can be solved (see [22, pp. 21–22]). On the other hand, if (q, q_Γ) is its unique solution, one shows that on can reconstruct p in order that the triplet (p, q, q_Γ) solves problem (2.25) – (2.27) (see [22, Thm. 5.4], in particular formulas [22, (5.10) – (5.11)]).

At this point, we let τ tend to zero in (2.34) – (2.36) rather than in (2.25) – (2.27). By doing that, we do not care about the limit of p^τ. To this end, we need some more tools. We introduce the spaces

$$\mathcal{W} := L^2(0,T;\mathcal{V}_\Omega) \cap \big(H^1(0,T;V^*) \times H^1(0,T;V_\Gamma^*)\big), \tag{2.37}$$

$$\mathcal{W}_0 := \{(v,v_\Gamma) \in \mathcal{W} : \ (v,v_\Gamma)(0) = (0,0)\} \tag{2.38}$$

and endow them with their natural topologies. Moreover, we denote by $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ the duality product between $\mathcal{W}_0^*$ and $\mathcal{W}_0$. We have the following representation result for the elements of the dual space $\mathcal{W}_0^*$ (see [23, Prop. 2.6]):

Proposition 1. *A functional $F : \mathcal{W}_0 \to \mathbb{R}$ belongs to $\mathcal{W}_0^*$ if and only if there exist Λ and Λ_Γ satisfying*

$$\Lambda \in \big(H^1(0,T;V^*) \cap L^2(0,T;V)\big)^* \quad \textit{and} \quad \Lambda_\Gamma \in \big(H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma)\big)^*, \tag{2.39}$$

$$\langle\!\langle F,(v,v_\Gamma)\rangle\!\rangle = \langle \Lambda, v\rangle_Q + \langle \Lambda_\Gamma, v_\Gamma\rangle_\Sigma \quad \textit{for every } (v,v_\Gamma) \in \mathcal{W}_0\,, \tag{2.40}$$

where the duality products $\langle\,\cdot\,,\,\cdot\,\rangle_Q$ and $\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma$ are related to the spaces X^* and X with $X = H^1(0,T;V^*) \cap L^2(0,T;V)$ and $X = H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma)$, respectively.

However, this representation is not unique, since different pairs $(\Lambda, \Lambda_\Gamma)$ satisfying (2.39) could generate the same functional F through formula (2.40).

At this point, we can state our last result. The following theorem gives both a generalized solution to a proper adjoint problem with $\tau = 0$ and a first-order necessary condition for optimality similar to (2.28) (see [23, Thm. 2.7]).

Theorem 9. *Assume* (1.1) – (1.6) *and* (1.9) – (1.12)*, and let $\mathcal{J}$ and $\mathcal{U}_{\mathrm{ad}}$ be defined by* (2.17) *and* (2.14) *under the assumptions* (2.15)*. Moreover, let $\overline{u}_\Gamma$ be any optimal control related to the state system with $\tau = 0$. Then, there exist Λ and Λ_Γ satisfying* (2.39)*, and a pair (q, q_Γ) satisfying*

$$q \in L^\infty(0,T;V^*) \cap L^2(0,T;V), \tag{2.41}$$

$$q_\Gamma \in L^\infty(0,T;H_\Gamma) \cap L^2(0,T;V_\Gamma), \tag{2.42}$$

$$(q,q_\Gamma)(t) \in \mathcal{V}_\Omega \quad \textit{for a.e. } t \in [0,T], \tag{2.43}$$

as well as

$$\int_0^T \langle \partial_t v, \mathcal{N} q\rangle + \int_0^T \langle \partial_t v_\Gamma, q_\Gamma\rangle_\Gamma + \int_Q \nabla q \cdot \nabla v + \int_\Sigma \nabla_\Gamma q_\Gamma \cdot \nabla_\Gamma v_\Gamma$$
$$+ \langle \Lambda, v\rangle_Q + \langle \Lambda_\Gamma, v_\Gamma\rangle_\Sigma = \int_Q \varphi_Q\, v + \int_\Sigma \varphi_\Sigma\, v_\Gamma \qquad \textit{for every } (v,v_\Gamma) \in \mathcal{W}_0\,, \tag{2.44}$$

such that

$$\int_\Sigma (q_\Gamma + b_0 \overline{u}_\Gamma)(v_\Gamma - \overline{u}_\Gamma) \ge 0 \quad \textit{for every } v_\Gamma \in \mathcal{U}_{\mathrm{ad}}. \tag{2.45}$$

Remark 4. In particular, if $b_0 > 0$, (2.45) says that

$$\overline{u}_\Gamma \ \textit{ is the orthogonal projection of } -q_\Gamma/b_0 \ \textit{ on } \mathcal{U}_{\mathrm{ad}} \tag{2.46}$$

with respect to the standard scalar product in $L^2(\Sigma)$.

One recognizes in (2.44) a problem that is analogous to (2.35) – (2.36). Indeed, if Λ, Λ_Γ and the solution (q, q_Γ) were regular functions, then its strong form should contain both a generalized backward parabolic equation like (2.35) and a final condition for $(\mathcal{N}q, q_\Gamma)$ of type (2.36), since the definition of $\mathcal{W}_0$ allows its elements to be free at $t = T$. However, the terms $f''(\overline{y}^\tau)q^\tau$ and $f''_\Gamma(\overline{y}^\tau_\Gamma)q^\tau_\Gamma$ are just replaced by the functionals Λ and Λ_Γ and cannot be identified as products, unfortunately.

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ПОСЛЕДНИЕ РЕЗУЛЬТАТЫ ДЛЯ УРАВНЕНИЯ КАНА – ХИЛЛИАРДА С ДИНАМИЧЕСКОЙ ГРАНИЦЕЙ

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В статье рассматривается уравнение Кана – Хиллиарда «чистое» или с вязкостью с возможно сингулярными потенциалами и динамическими граничными условиями. Обсуждается корректность соответствующей начальной задачи. Изучается задача граничного управления для системы Кана – Хиллиарда с вязкостью и находятся необходимые условия оптимальности первого порядка. Кроме того, ставится аналогичная задача граничного управления для «чистой» системы Кана – Хиллиарда, результаты получаются помощью предельного перехода в случае системы Кана – Хиллиарда с вязкостью, когда коэффициент вязкости стремится к нулю.

Ключевые слова: уравнение Кана – Хиллиарда; динамические граничные условия; разделение фаз; корректность; оптимальное граничное управление; условия оптимальности.

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