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THE CONTRIBUTION OF ANGELO FAVINI IN TWENTY YEARS OF JOINT RESEARCH (1996 – 2016)

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During his long activity in research, Angelo Favini has given several relevant contributions in many fields of mathematics. In our long collaboration, testified by more than thirty joint papers, I highly appreciated his deep competence, his brilliant intuitions, his extraordinary knowledge of the contemporary literature, in addition to a special acumen in finding possible critical points to be clarified, or better solved, during the preparation of the papers.

Here follows a short survey of the main problems studied in our joint papers over the last twenty years, often in collaboration with other outstanding mathematicians.

(1) At the beginning of 1990s there was a long standing open problem concerning the existence of analytic semigroups generated by second order elliptic differential operators degenerating at the boundary. It is well-known that the existence of a (C_0) semigroup guarantees the wellposedness of the abstract Cauchy problem associated with the evolution equation governed by the generator, provided that the initial datum is in its domain. The analyticity of the semigroup provides, in addition, the best possible regularity of the solution, even by starting from the initial datum in the ambient space. As already known in Probability, or in Approximation Theory, in the space of continuous functions $C(\overline{\Omega})$ (here Ω is a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$), a possibly degenerate on the boundary, second order elliptic operator A can be naturally equipped with the so-called Wentzell boundary condition $Au = 0$, introduced in the one-dimensional case by W. Feller in his pioneer work [7] (see also [6]), and in the multidimensional case by A.D. Wentzell [38]. For Wentzell boundary conditions in different spaces see e.g. [33, 37]. Observe that, in an evolution equation $u_t = Au$, replacing Au by u_t in Wentzell boundary condition reveals that, under suitable regularity assumptions on the elements of the domain of A, a solution u of the equation with Wentzell boundary condition remains constant with respect to t along the boundary. An easy example of degenerate elliptic second order differential operator on the space $C[0,1]$ equipped with Wentzell boundary conditions is $A_j u = x^j(1-x)^j u''$, $j > 0$, well-known also for governing some evolution problems in genetics. In [1] the existence of analytic semigroups generated by A_j was proved for $j \geq 2$ on $C[0,1]$ by using different types of boundary conditions, including Wentzell's ones. After that, much attention was paid to this problem and relevant contributions by A. Favini et al. appeared in different papers, where, in addition to the space of continuous functions, as in [27, 30, 31], also L^p spaces, with or without weights, and Sobolev spaces were considered, see e.g. [2, 20, 28, 29]. Results concerning the wave equation with Wentzell boundary conditions in $C[0,1]$ were also obtained in [16].

(2) In the real-valued space $C(\overline{\Omega})$ other interesting problems concern the existence of Feller semigroups generated by second-order elliptic differential operators degenerating on $\partial\Omega$ and equipped with different boundary conditions, including Wentzell's ones. In this framework, some enlightening results were obtained by K. Taira, A. Favini and S. Romanelli in the papers [34–36], under suitable regularity assumptions on the coefficients of the operator and on $\partial\Omega$.

(3) If A denotes a second order, linear or nonlinear, elliptic differential operator, a general boundary condition including both Robin (and Dirichlet, Neumann) boundary conditions $b\frac{\partial u}{\partial n}+cu=0$, relevant for L^p spaces, and Wentzell boundary conditions $Au=0$, relevant for spaces of continuous functions, arises naturally as general Wentzell boundary condition given by

$$(GWBC) \quad aAu(x)+b\frac{\partial u(x)}{\partial n}+cu(x)=0, \quad x\in\partial\Omega,$$

where $(a,b,c)\neq(0,0,0)$ and $\frac{\partial u}{\partial n}$ is the outer normal derivative of u. In [13] A. Favini, G.R. Goldstein, J.A. Goldstein and S. Romanelli introduced *(GWBC)* for the operator $Au=\alpha u''$ with domain

$$D(A)=\{u\in C[0,1]\cap C^2(0,1):\ Au\in C[0,1],\ u\,\text{satisfies}\,(GWBC)\}$$

in $C[0,1]$. Note that, in this case, *(GWBC)* consists of two conditions and reads as

$$a_jAu(j)+(-1)^jb_ju'(j)+c_ju(j)=0, \quad j=0,1,$$

with a_j,b_j,c_j real numbers.

Therein, under the assumptions that $\alpha\in C(0,1)$, $\alpha>0$ and $\frac{1}{\alpha}\in L^1(0,1)$ (α can vanish at $\{0,1\}$) and suitable assumptions on a_j,b_j,c_j, $j=0,1$, the authors stated sufficient conditions yielding meaningful properties of the operator A in $C[0,1]$, as density of the domain, closedness, dissipativity and range condition. For instance, A is densely defined and m-dissipative whenever $a_0=1=a_1$ and $(-1)^jb_j\leq 0$ and $c_j\geq 0$, with $c_j=0$ if $b_j=0$, $j=0,1$. More general nonlinear operators and nonlinear *(GWBC)* can also be considered (see e.g. [17]). Hence, the previous results provide extremely useful tools in the study of wellposedness for abstract Cauchy problems associated with a wide class of evolution problems.

(4) The wellposedness of abstract Cauchy problems with *(GWBC)* in L^p spaces, $1\leq p<\infty$ requires the introduction of special spaces, as, for instance, discussed in the simple case of $Au=u''$, acting on $L^2(0,1)$, equipped with *(GWBC)*, see [18, Section 5]. Hence, in order to study (C_0)-semigroups governed by operators equipped with *(GWBC)* in spaces of L^p-type, a new approach is necessary. In [18] A. Favini, J.A. Goldstein, G.R. Goldstein and S. Romanelli assumed that A is the elliptic second order differential operator in divergence form $Au=\nabla\cdot(a\nabla)$, where $a\in C^1(\overline{\Omega})$, a is nonnegative and satisfies $\Gamma:=\{x\in\partial\Omega:\ a(x)>0\}\neq\emptyset$, and A is equipped with the following *(GWBC)*

$$Au+\beta\frac{\partial u}{\partial n}+\gamma u=0 \quad on \quad \Gamma,$$

with β,γ nonnegative functions in $C^1(\partial\Omega)$, $\beta>0$. Note that, if we consider the evolution equation $u_t=Au$ equipped with *(GWBC)*, under assumptions of sufficient regularity for u, then we can plug the term $u_t=Au$ in the boundary condition and obtain the condition $u_t+\beta\frac{\partial u}{\partial n}+\gamma u=0$. Hence, the term Au corresponds to introduce a dynamic condition on the boundary. Coming back to the introduction of the new spaces, let us assume that $\Gamma=\partial\Omega$. For $1\leq p<\infty$, the correct L^p space to consider is $X_p:=L^p(\overline{\Omega},d\mu)$, where $d\mu=dx|_\Omega\oplus\frac{adS}{\beta}|_{\partial\Omega}$. Here dx denotes the Lebesgue measure on Ω and $\frac{adS}{\beta}$ denotes the natural surface measure dS on $\partial\Omega$ with weight $\frac{a}{\beta}$. For $p=\infty$ the space X_∞ can be identified with $C(\overline{\Omega})$. One can interpret X_p, $1 \leq p < \infty$, as the completion of $C(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{X_p}$ given by $\|U\|_{X_p} = \left(\int_\Omega |u|^p\, dx + \int_{\partial\Omega} |u|^p \frac{adS}{\beta}\right)^{\frac{1}{p}}$, where, if $u \in C(\overline{\Omega})$, we consider $U = (u|_\Omega, u|_{\partial\Omega})$. In general, a member of X_p is a pair $H = (f, g)$, where $f \in L^p(\Omega, dx)$ and $g \in L^p(\partial\Omega, \frac{adS}{\beta})$ and, for $p < \infty$, f may not have a trace on $\partial\Omega$, and even if f does, this trace needs not equal g. For $p = 2$, X_2 is a Hilbert space equipped with the inner product $< H_1, H_2 >_{X_2} = < f_1, f_2 >_{L^2(\Omega)} + < g_1, g_2 >_{L^2(\partial\Omega, \frac{adS}{\beta})}$, with $H_i = (f_i, g_i) \in X_2$, $i = 1, 2$. Under previous assumptions, in [18], the authors proved that there exists a (C_0) semigroup generated by the closure of the realization of A in X_p, $1 \leq p \leq \infty$, and this semigroup is analytic if $1 < p < \infty$. In addition, A is essentially self-adjoint on X_2. Existence and analyticity results for the semigroup associated with A remain true also for more general elliptic versions of A, equipped with the corresponding $(GWBC)$. See [11]. It is very important to point out that there is a precise physical derivation of *(GWBC),* as it was shown in details by G.R. Goldstein in [32]. For instance, in the case of the heat equation, in the evaluation of the total heat content of the region, the use of *(GWBC)* allows to consider also the contribution of a heat source located on the boundary, while in all usual approaches concerning the traditional boundary conditions, this type of contribution appears completely neglected. All the above results opened the way to a more and more wide literature and to a diffused interest of the international community of mathematicians, including those ones involved in the study of dynamical boundary conditions. Thus, in my opinion, the above results represent the milestone of our scientific collaboration.

(5) In the last years, starting from the paper [11], A. Favini, G.R. Goldstein, J.A. Goldstein, E. Obrecht and S. Romanelli found a more complete formulation of *(GWBC)* in this form

$$(GGWBC) \quad Au + \beta\frac{a\partial u}{\partial n} + \gamma u - q\beta\Delta_{LB} u = 0, \quad on \quad \partial\Omega,$$

including the term with the Laplace – Beltrami operator Δ_{LB} having nonnegative constant coefficient q. In [11, 12] the authors assumed that A is a uniformly elliptic operator in divergence form of general type. Here we refer to A given as in (4) and to the assumptions on the coefficients in *(GGWBC)* as before. Then the realization of A in X_2, with a suitable domain, including $(GGWBC)$, has a dissipative self-adjoint closure which generates an analytic semigroup in X_2 with angle of analyticity $\frac{\pi}{2}$. In the same papers [11, 12], under C^∞ regularity assumptions on $\partial\Omega$, on all the coefficients of A and on β, γ, we also showed that the closure of the realization of A in X_p, $1 < p < \infty$, is m-dissipative and generates an analytic semigroup having sector of analyticity depending on the moduli of ellipticity of A. Moreover, its domain can be explicitly described. In a long series of subsequent papers by A. Favini, G.R. Goldstein, J.A. Goldstein, S. Romanelli and other coauthors, as G.M. Coclite, C.G. Gal and E. Obrecht, see e.g. [3–5, 8, 12, 21, 25], additional interesting results were obtained with respect to continuous dependence of the solutions from the boundary coefficients, hyperbolic problems, nonautonomous wave equation, nonlinear operators, nonsymmetric operators, unbounded domains.

(6) Extensions of some previous results described in (5) can be obtained also in the case of some classes of higher order operators acting on X_2. See e.g. [22, 24]. In these papers Wentzell boundary conditions together with lower order boundary conditions are

associated with this type of operators. For instance, in $X_2 = L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{dS}{\beta})$, a fourth order operator of the type $Au = \Delta(a\Delta)u$, equipped with both boundary conditions $Au + \beta\frac{\partial(a\Delta u)}{\partial n} + \gamma u = 0$ and $\Delta u = 0$ is essentially self-adjoint and bounded below, provided that $a \in C^4(\overline{\Omega})$, $a > 0$, $\partial\Omega$ is C^4 and $\beta, \gamma \in C^{3+\epsilon}(\partial\Omega)$. A more complete classification of $(GWBC)$ for the fourth order differential operator $Au = u''''$ acting on X_2, with $\Omega = (0,1)$, can be found in [23] where we obtained a characterization of the symmetry and sufficient conditions for self-adjointness and quasi-m-accretivity of A.

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