ОБЗОРНЫЕ СТАТЬИ

MSC 34A09, 65N99 **DOI: 10.14529/mmp170201**

ON THE CONCEPT OF INDEX FOR PARTIAL DIFFERENTIAL ALGEBRAIC EQUATIONS ARISING IN MODELLING OF PROCESSES IN POWER PLANTS

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This paper addresses some classes of linear and quasi-linear partial differential algebraic equations (PDAEs), i.e. systems of partial differential equations with singular matrices multiplying the higher derivatives of the desired vector-function. Such systems do not belong to the class of the Cauchy – Kovalevskaya equations, and therefore do not not comply with known existence theorems. The current research focuses on the first order evolutionary systems with one variable and investigates PDAEs depending on the parameter. The concept of index for PDAEs is introduced and various statements of initial boundary problems are considered. The results obtained are used to simulate and analyze the heat and mass exchange processes in power plants.

Keywords: partial differential algebraic equation; partial derivatives; integral differential equations; solution space; index.

Introduction and Statement of the Problem

Consider an evolutionary system of partial differential equations

$$\Lambda(D_t, D_x)u := A(x,t)D_t u + \sum_{j=1}^{\rho} B_j(x,t)D_x^j u + C(x,t)u = f(x,t),\ (x,t) \in \mathbf{U}, \quad (1)$$

where $A(x,t),\ B_j(x,t),\ C(x,t)$ are $(n \times n)$-matrices, $\mathbf{U} = X \times T \subseteq \mathbf{R}^2$, $X = [x_0, x_1]$, $T = [t_0, t_1]$, $D_t \equiv \partial/\partial t,\ D_x \equiv \partial/\partial x,\ f(x,t),\ u \equiv u(x,t)$ are the given and the desired vector-functions, respectively. It is assumed that

$$\det\ A(x,t) = 0,\ \det\ B_\rho(x,t) = 0\ \forall (x,t) \in \mathbf{U}, \quad (2)$$

and that the entries of (1) are sufficiently smooth in some domain $\tilde{\mathbf{U}}$ that includes $\mathbf{U}$. The solution $u(x,t)$ is searched for in the domain $\mathbf{U}$. In this paper, we focus only on classic solutions.

In what follows, by the solution of (1) we understand any vector-function $u(x,t)$ that has continuous partial derivatives in $\tilde{\mathbf{U}}$ with respect to x, t and turns (1) into an identical relation in $\mathbf{U}$.

The statement of the problem for partial differential equations usually includes initial and boundary conditions. Here we consider the simplest cases:

$$\mathrm{u}_j(x,t_0) = \phi_j(x),\ u(x_0,t) = \psi(t),\ \mathrm{u}_j(x,t) = D_x^j u(x,t),\ j = \overline{0,\rho},\ D_x^0 u(x,t) = u(x,t). \quad (3)$$

Ever since the second half of the 20th century the field of mathematics addressing equations with a noninvertible operator at the evolutionary term has played an important role in various applications such as hydrodynamics (the Navier-Stokes equations), gas dynamics (the Euler equations), electric and thermal engineering [1–8].

The study of such equations began with the work by L.S. Sobolev [1], that is why they are often referred to as Sobolev equations [2]. It is quite common to treat such equations by transition to the differential equations in the Banach spaces

$$\mathbf{A}\dot{v}(t) + \mathbf{B}v(t) = \mathbf{f}(t), \ t \in T, \tag{4}$$

where $\mathbf{A}, \mathbf{B}$ are some operators that put (4) into correspondence to (1) in the Banach spaces, ker $\mathbf{A} \neq 0$; and $v(t)$, $\mathbf{f}(t)$ are the desired and the given vector-functions, correspondingly.

A significant contribution into this field of mathematics has been made by G.A. Sviridyuk and his followers (see, for example, [2–7] and the references listed there). Interesting results are also presented in [9–14]. Another approach to solving Sobolev equations suggests transition to singular in some sense partial differential equations with subsequent application of powerful methods of functional analysis [15, 16]. Some promising results have been obtained for systems (1) with constant coefficient matrices by employment of Furrier transformations and similar methods (see, for example, the fundamental monographs [17, 18] and the references listed there).

Finally, during the last 15–20 years it has become popular to employ the approach based on the methods developed for the DAE theory [19–25]. According to the American Mathematical Society, the term DAE is used for systems of ordinary differential equations with a singular matrix multiplying the higher derivative of the desired vector-function. Index is a notion used in the theory of DAEs for measuring the distance from a DAE to its related ODE. The index is a nonnegative integer number that provides useful information about the mathematical structure and potential complications in the analysis and the numerical solution of the DAE. It also identifies the number of derivatives on which the solution to the given DAE depends. However, there is still no agreement on how to calculate the index of partial differential algebraic equations (PDAEs), and the current research aims to provide some clarity on this matter. We will address a special case of (1) that comprises partial differential equations, ordinary differential equations, and algebraic equations.

When studying PDAEs, we face the question whether we can classify them as hyperbolic, elliptic, or parabolic, because the classic theory of partial differential equations states that the type of the system predetermines the method of solution (see, for example, [26]). Therefore, in what follows, we say that a PDAE is hyperbolic if it can be split into: 1) a classic hyperbolic system; 2) differential subsystems with respect to x, t, where the second variable is treated as a parameter; 3) a subsystem with a unique solution, in particular, an algebraic system.

Remark 1. For the sake of simplicity, the dependence on t and x sometimes will be omitted, if this does not lead to misunderstanding. The inclusion $V(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$, $i, j > 1$, where $V(x,t)$ is some matrix (in particular, a vector-function), denotes that all elements of $V(x,t)$ have continuous partial derivatives up to orders i, j in the domain $\mathbf{U}$. If $i = j$, then we say that the matrix $V(x,t)$ is i times differentiable in the domain $\mathbf{U}$. $V_1(x) \in \mathbf{C}^i(X)$, $V_2(t) \in \mathbf{C}^i(T)$ denote i-times differentiable matrices $V_1(x)$, $V_2(t)$. The continuous matrices are denoted as $V(x,t) \in \mathbf{C}(\mathbf{U}),\ V_1(x) \in \mathbf{C}(X),\ V_2(t) \in \mathbf{C}(T)$ and $\mathbf{r}[V(x,t)] = \max\{\operatorname{rank} V(x,t),\ (x,t) \in \mathbf{U}\}$.

Now consider an example to illustrate some properties specific to PDAEs.

Example 1.

$$\Lambda(D_t, D_x)u = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} D_t u +$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{xt} & 1 \\ 0 & 0 & 0 \end{pmatrix} D_x^2 u + \begin{pmatrix} \alpha_3 & \alpha_4 & \alpha_5 \\ 0 & e^{xt} + 2t & 1 \\ 0 & 0 & 0 \end{pmatrix} D_x u + \begin{pmatrix} \alpha_6 & \alpha_7 & \alpha_8 \\ 0 & \gamma & 0 \\ 0 & e^{xt} & \delta \end{pmatrix} u = f.$$

Here $u = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^\top$, $f = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}^\top$, $\delta,\ \alpha_i,\ i = \overline{1,8}$ are numeric parameters, $\gamma \equiv \gamma(x,t)$ is some smooth function, $\top$ stands for transposition. However, in this situation, if $\delta = 1$ and $\gamma \equiv 0$, the system is solvable for any $f_1 \in \mathbf{C}^{1,1}(\mathbf{U})$, $f_2 \in \mathbf{C}^{1,1}(\mathbf{U})$, $f_3 \in \mathbf{C}^{3,1}(\mathbf{U}), \gamma \in \mathbf{C}^{1,1}(\mathbf{U})$, if $g(x,t) = [\gamma(x,t) - (t+t^2)e^{xt}] \neq 0\ \forall (x,t) \in \mathbf{U}$. Indeed, the third equation of the system yields $u_3 = f_3 - e^{xt}u_2$. Substitute u_3 into the second equation. We obtain $u_2 = (f_2 - D_x f_3 - D_x^2 f_3)/g(x,t)$. Therefore, the components $u_2,\ u_3$ are uniquely defined in the domain $\mathbf{U}$ and belong to $\mathbf{C}^{1,1}(\mathbf{U})$. Then, by substituting $u_2,\ u_3$ into the first equation, we obtain an equation of the hyperbolic type

$$D_t u_1 + \alpha_3 D_x u_1 + \alpha_6 u_1 = \psi(x,t) f_1 - \alpha_3 u_2 - \alpha_4 u_3,$$

where $\psi(x,t) = f_1 - \alpha_1 D_t u_2 - \alpha_2 D_t u_3 - \alpha_4 D_x u_3 - \alpha_5 D_x u_3 - \alpha_7 u_2 - \alpha_8 u_3$. Hence, we can say that the system is implicitly hyperbolic and the following equality is valid

$$u_1 = \phi(x - \alpha_1(t - t_0)) + \int\limits_{t_0}^{t} \exp(\alpha_3 s)\psi(x - \alpha_3(t-s), s)ds,$$

where $\phi(z)$ is an arbitrary function.

Summarizing what has been said, we are drawn to the following conclusions:

1) the components $u_2,\ u_3$ are fixed functions. Hence, we can set initial and boundary conditions only in the form of the functions $u_2(x,t_0),\ u_2(x_0,t),\ u_3(x,t_0),\ u_3(x_0,t)$;

2) the equation is hyperbolic with respect to u_1, and here we can set arbitrary initial and boundary conditions $u_1(x_0,t), u_1(x,t_0)$ that satisfy the consistency conditions at the point (x_0,t_0). For example, $\phi(x_0) = \psi(t_0)$ etc. [26];

3) if we perturb the free term $\tilde{f_3} = f_3 + \epsilon \sin(tx/\epsilon^2)$, then it can be readily seen that at $\epsilon \to 0$ the following relations are valid: $\|\tilde{f_3} - f_3\|_{\mathbf{C}(\mathbf{U})} \to 0,\ \|\tilde{u}_2 - u_2\|_{\mathbf{C}(\mathbf{U})} \to \infty$, which means that the solution is highly sensitive to changes in the initial data.

1. Auxiliary Information

Definition 1. [27] *A pseudo inverse of the $(m \times n)$-matrix $M(x,t)$, $t \in \mathbf{U}$ is defined as an $(n \times m)$-matrix $M^+(x,t)$ satisfying the following criteria*

$$M(x,t)M^+(x,t)M(x,t) = M(x,t),\ M^+(x,t)M(x,t)M^+(x,t) = M^+(x,t),$$

$$(M^+(x,t)M(x,t))^\top = M^+(x,t)M(x,t),\ (M(x,t)M^+(x,t))^\top = M(x,t)M^+(x,t).$$

$M^+(x,t)$ exists for any matrix and any $(x,t) \in \mathbf{U}$. If the matrix $M(x,t)$ is square and regular, then $M^{-1}(x,t) = M^+(x,t)$, and $M^{-1}(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$, if $M(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$.

Lemma 1. *Let* $M(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$ *and* $\mathrm{rank} M(x,t) = \mathrm{const} = r\ \forall (x,t) \in \mathbf{U}$.
Then:

1. *There exist square matrices* $L(x,t)$, $R(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$ *such that* $\det\ L(x,t) \neq 0$, $\det\ R(x,t) \neq 0\ \forall (x,t) \in \mathbf{U}$, $L(x,t)M(x,t)R(x,t) = \mathrm{diag}\{I_r, 0\}$, *where* I_ν *is an identity matrix of dimension* ν;
2. *There exists the matrix* $M^+(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$.

If rank $M(x,t) \neq \mathrm{const}$, $(x,t) \in \mathbf{U}$, then at least one element of $M^+(x,t)$ has a discontinuity of the second kind in the domain $\mathbf{U}$. The proof techniques can be found in the monograph [28].

Now consider a higher order DAE depending on a parameter

$$\Lambda_k(D_t)u := \sum_{j=0}^{k} A_j(x,t) D_t^j u = f(x,t), \tag{5}$$

$$\Lambda_k(D_x)u := \sum_{j=0}^{k} A_j(x,t) D_x^j u = f(x,t), \tag{6}$$

where $(x,t) \in \mathbf{U}$, $A_j(x,t)$ are $(n \times n)$-matrices at least from $\mathbf{C}(\mathbf{U})$, $\det A_k(x,t) \equiv 0$, the variables x and t are understood as parameters. Introduce the following notation.

Definition 2. *The operator* $\Omega_l(D_t) := \sum_{j=0}^{l} L_j(x,t) D_t^j$, *where* $L_j(x,t)$ *are* $(n\times n)$*-matrices from* $\mathbf{C}(\mathbf{U})$, *with the property*

$$\Omega_l(D_t) \circ \Lambda_k(D_t) y = \sum_{j=0}^{k} \tilde{A}_j(x,t) D_t^j y\ \forall y \in \mathbf{C}^{k+1}(\mathbf{U}),\ \det\ \tilde{A}_k(x,t) \neq 0\ \forall (x,t) \in \mathbf{U},$$

is called the Left Regularizing Operator (LRO) for the DAE (5). The smallest possible number l *is said to be the index of (5).*

A similar definition of the LRO can be formulated for (6) by replacing D_t with D_x.

Lemma 2. *If system (5) has index* l*, then the following alternative holds:* $\det A_k(x,t) \neq 0\ \forall (x,t) \in \mathbf{U}$ *for* $l = 0$*, or* $\det A_k(x,t) \equiv 0$, $(x,t) \in \mathbf{U}$ *for* $l > 0$.

Proof. Indeed, if $l = 0$, then $L_0 A_k = \tilde{A}_k\ \forall (x,t) \in \mathbf{U}$, where $\det\ \tilde{A}_k \neq 0\ \forall (x,t) \in \mathbf{U}$. However, if $l > 0$, then it follows from the definition of index that $L_l A_k = 0\ \forall (x,t) \in \mathbf{U}$. This is valid for continuous matrices L_l and A_k if and only if $\det\ L_l = \det\ A_k = 0\ \forall (x,t) \in \mathbf{U}$.

□

In other words, if we assume that the LRO exists, then the condition $\det A_k(x,t) \equiv 0$, $(x,t) \in \mathbf{U}$ is not necessary. Moreover, the LRO guarantees solvability of the system for any fixed x.

Theorem 1. *Let system (5) satisfy the conditions:*

1. $A_j(x,t) \in \mathbf{C}^{m,i}(\mathbf{U}),\ j=\overline{0,k},\ m=\max\{(k-1)n+r+1, 2l\},\ r=\mathbf{r}[A_k(x,t)],$ $i \geq 0,\ f(x,t) \in \mathbf{C}^{l,i}(\mathbf{U})$*;*
2. *The system has the LRO in* $\mathbf{U}$*, which coefficients are either continuous or* i *times partially differentiable with respect to* x*.*

Then, system (5) is solvable for any $f(x,t)$*, and its solution for any fixed* $x \in X$ *can be written in the form*

$$u(x,t) = X_d(x,t)c(x) + W(D_t)f(x,t),$$
$$W(D_t)f(x,t) = \int\limits_{t_0}^{t} K(x,t,s)f(x,s)ds + \sum_{j=0}^{l-k} C_j(x,t)D_t^j f, \tag{7}$$

where $X_d(x,t)$ *is an* $(n \times d(x))$*-matrix,* $K(x,t,s)$*,* $C_j(x,t)$ *are* $(n\times n)$*-matrices smooth with respect to* t*,* $j=\overline{0,l-1}$*,* $\operatorname{rank} X_d(x,t) = d(x)\ \forall t \in T$*,* $c(x)$ *is an arbitrary function. If* $c(x) \in \mathbf{C}^i(X)$*, then* $u(x,t) \in \mathbf{C}^{k,i}(\mathbf{U})$*.*

If $l<k$*, the vector-function in (7) has the form* $W(D_t)f(x,t) = \int\limits_{t_0}^{t} K(x,t,s)f(x,s)ds$*.*

Proof. Denote $\zeta = \begin{pmatrix} u^\top & D_t u^\top & \dots & D_t^{(k-1)}u^\top \end{pmatrix}^\top$. Then we can put the following first order DAE into correspondence to (5):

$$\begin{pmatrix} I_\nu & 0 \\ 0 & A_k(x,t) \end{pmatrix} D_t\zeta + \begin{pmatrix} 0 & -I_\nu \\ A_0(x,t) & \tilde{A}(x,t) \end{pmatrix}\zeta = \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix},\ (x,t) \in \mathbf{U}, \tag{8}$$

where $\nu=(k-1)n$, $\tilde{A} = \begin{pmatrix} A_1 & A_2 & \dots & A_{k-1} \end{pmatrix}$. System (5) has the LRO of the form $\operatorname{diag}\{I_\nu, \Omega_l(D_t)\}$, and the proof is based on application of the statement that was proved in [29] for the situation when $k=1$ and the coefficient matrices as well the free term depend on t only. Note that all solutions to (5) are the solutions to the non-singular system

$$\Omega_l(D_t) \circ \Lambda_k(D_t)y = \Omega_l(D_t)f(x,t),\ (x,t) \in \mathbf{U}. \tag{9}$$

If in Definition 1 matrices $\tilde{A}_j(x,t)$ and the vector-function $\Omega_l f(x,t)$ are either continuous or i-times differentiable with respect to x, then any solution $y \equiv y(x,t)$ to (9) is either continuous or i-times differentiable with respect to x. Hence, the solutions to (5) possess the same properties.

□

A similar theorem can be formulated for the DAE (6). Then, according to Theorem 1, the general solution to (6) can be written in the form of the equalities:

$$u = X_d(x,t)c(t) + W(D_x)f(x,t),$$
$$W(D_x)f(x,t) = \int\limits_{x_0}^{x} K(x,t,s)f(s,t)ds + \sum_{j=0}^{l-k} C_j(x,t)D_x^j f. \tag{10}$$

2. Index for Linear PDAEs

Now, using the results from the previous section, introduce the concept of index for PDAEs.

Definition 3. *Let there exist an operator* $\Psi_l(D_t, D_x) := \sum_{j=0}^{l} \mathcal{L}_j(D_x) D_t^j$, *where* $\mathcal{L}_j(D_x) = \sum_{i=0}^{\sigma_j} L_i(x,t) D_x^i$, $L_i(x,t)$ *are* $(n \times n)$*-matrices from* $\mathbf{C}(\mathbf{U})$, *with the property*

$$\Psi_l(D_t, D_x) \circ \Lambda(D_t, D_x) y = \mathcal{A}(D_x) D_t y + \tilde{\mathcal{A}}(D_x) y \ \forall y \in \mathbf{C}^{l+1}(\mathbf{U}),$$

where

$$\mathcal{A}(D_x) = \sum_{i=0}^{m} \mathcal{A}_i(x,t) D_x^i, \ \tilde{\mathcal{A}}(D_x) = \sum_{i=0}^{m_1} \tilde{\mathcal{A}}_i(x,t) D_x^i,$$

$\mathcal{A}_i(x,t)$, $\tilde{\mathcal{A}}_i(x,t)$ *are* $(n \times n)$*-matrices from* $\mathbf{C}(\mathbf{U})$*; it is assumed that the operator* $\mathcal{A}(D_x)$ *has the LRO in the domain* $\mathbf{U}$*. The smallest possible* l *is said to be the index of system (1) with respect to the variable* t*.*

Due to the fact that the partial derivatives with respect to x and t play equally important roles in the system, the index with respect to x can be defined in a similar way. If system (1) has index with respect to t, then, using formulas (10) and provided that the initial data is sufficiently smooth, the original system can be reduced to a vector integral differential equation resolved with respect to the evolutionary term

$$D_t u + W(D_x) \tilde{\mathcal{A}}(D_x) u = W(D_x) f(x,t) + X_d(x,t) c(t). \tag{11}$$

Now suppose that the conditions of Lemma 1 are satisfied. Then, by multiplying system (9) by $L(x,t)$ on the left and introducing the change of variable $u = R(x,t)z$, we obtain

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} D_t z + \begin{pmatrix} \Lambda_1^{11}(D_x) & \Lambda_1^{12}(D_x) \\ \Lambda_1^{21}(D_x) & \Lambda_1^{22}(D_x) \end{pmatrix} z = g, \tag{12}$$

where $g = L(x,t)f$, $\Lambda_1^{i\nu}(D_x) = \sum_{j=1}^{\rho} \tilde{B}_{i\nu}(x,t) D_x^j + \tilde{C}_{i\nu}(x,t)$, $\tilde{B}_{i\nu}(x,t)$, $\tilde{C}_{i\nu}(x,t)$, $i, \nu = 1, 2$ are the blocks of the matrices $LB_\rho D_x^\rho R$, $\cdots, L\sum_{j=1}^{\rho} jB_j D_x^j R$ and $LAD_t R + L\sum_{j=1}^{\rho} B_j D_x^j R + LCR$, correspondingly.

Example 2. Set in Example 1 $\alpha_1 = 0$, $\alpha_2 = 0$. Then the system has the form of the relation (12), where

$$\Lambda_1^{11}(D_x) = \alpha_3 D_x + \alpha_6, \ \Lambda_1^{12}(D_x) = (\alpha_4 D_x + \alpha_7 \quad \alpha_5 D_x + \alpha_8), \ \Lambda_1^{21}(D_x) = 0,$$

$$\Lambda_1^{22}(D_x) = \begin{pmatrix} e^{xt} & 1 \\ 0 & 0 \end{pmatrix} D_x^2 + \begin{pmatrix} e^{xt} + 2t & 1 \\ 0 & 0 \end{pmatrix} D_x + \begin{pmatrix} \gamma(x,t) & 0 \\ e^{xt} & 1 \end{pmatrix}.$$

Differentiate the second and the third equations of the system with respect to t. We get

$$\begin{pmatrix} 1 & 0 \\ 0 & \Lambda_1^{22}(D_x) \end{pmatrix} D_t u + \begin{pmatrix} \Lambda_1^{11}(D_x) & \Lambda_1^{12}(D_x) \\ 0 & \tilde{\Lambda}_1^{22}(D_x) \end{pmatrix} u = f, \tag{13}$$

where $\tilde{\Lambda}_1^{22}(D_x) = \begin{pmatrix} xe^{xt} & 0 \\ 0 & 0 \end{pmatrix} D_x^2 + \begin{pmatrix} xe^{xt} + 2 & 0 \\ 0 & 0 \end{pmatrix} D_x + \begin{pmatrix} D_t\gamma(x,t) & 0 \\ xe^{xt} & 0 \end{pmatrix}$. The operator $\Lambda_1^{22}(D_x)$ is index 2, if $g(x,t) = [\gamma(x,t) - (t + t^2)e^{xt}] \neq 0 \ \forall (x,t) \in \mathbf{U}$. Here the LRO has the form $\Omega_2(D_x) = \begin{pmatrix} 1 & 0 \\ -D_x & D_x^2 \end{pmatrix}$. Moreover, in formula (10) we have that $d = 0$ and $W(D_x) = \begin{pmatrix} e^{xt} & 1 \\ g(x,t) & 0 \end{pmatrix}^{-1} \Omega_2(D_x)$. Therefore, system (11) in this case has the form

$$D_t u + \begin{pmatrix} \Lambda_1^{11}(D_x) & \Lambda_1^{12}(D_x) \\ 0 & W(D_x)\tilde{\Lambda}_1^{22}(D_x) \end{pmatrix} u = \begin{pmatrix} f_1 \\ W(D_x)D_t f_2 \end{pmatrix}, \; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Theorem 2. *Let one of the following conditions be satisfied for system (12):*

1. *The operator $\Lambda_1^{22}(D_x)$ has the LRO in the domain;*
2. *$\Lambda_1^{22}(D_x) = 0$ and the operator $\Lambda_1^{21}(D_x) \circ \Lambda_1^{12}(D_x)$ has the LRO in the domain* $\mathbf{U}$.

Then: 1) under conditions of Theorem 1 the DAE (1) has index 1 with respect to t in the domain $\mathbf{U}$*; 2) if condition 2 of Theorem 1 is satisfied, then the DAE (1) has index 2 with respect to t in the domain* $\mathbf{U}$.

Proof. Transform the DAE (1) to the form (12). Differentiate the second block equation of (12) with respect to t. We obtain

$$\begin{pmatrix} I_r & 0 \\ \Lambda_1^{21}(D_x) & \Lambda_1^{22}(D_x) \end{pmatrix} D_t z + \begin{pmatrix} \Lambda_1^{11}(D_x) & \Lambda_1^{12}(D_x) \\ \tilde{\Lambda}_1^{21}(D_x) & \tilde{\Lambda}_1^{22}(D_x) \end{pmatrix} z = \tilde{g} = \begin{pmatrix} g_1 \\ D_t g_2 \end{pmatrix}, \; g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad (14)$$

where $\tilde{\Lambda}_1^{2j}(D_x) = D_t B_{2j}(x,t) D_x + D_t C_{2j}(x,t), \; j = 1, 2$.

Multiply the first block equation of (12) by the operator $\Lambda_1^{21}(D_x)$ and deduct the result from the second equation. This yields a system $\mathcal{A}(D_x) D_t z + \tilde{\mathcal{A}}(D_x) z = \tilde{g}$, where the operators $\mathcal{A}(D_x)$ and $\Psi_l(D_t, D_x)$ from Definition 3 can be written in the form $\mathcal{A}(D_x) = \operatorname{diag}\{I_r, \; \Lambda_1^{22}(D_x)\}$, $\Psi_l(D_t, D_x) = \begin{pmatrix} I_r & 0 \\ -\Lambda_1^{21}(D_x) & I_{n-r} \end{pmatrix} \mathcal{D}_t L(x,t)$, $\mathcal{D}_t = \operatorname{diag}\{I_r, \; D_t I_{n-r}\}$.

Let the second part of the statement be satisfied. If we again differentiate the second block equation of (12) with respect to t and deduct the first equation, multiplied by $\Lambda_1^{21}(D_x)$, from the second one, we arrive at

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} D_t z + \begin{pmatrix} \Lambda_1^{11}(D_x) & \Lambda_1^{12}(D_x) \\ \Phi(D_x) & -\Lambda_1^{21}(D_x)\Lambda_1^{12}(D_x) \end{pmatrix} z = h, \quad (15)$$

where $h = \begin{pmatrix} g_1^\top & D_t g_2^\top - \Lambda_1^{21}(D_x) g_1^\top \end{pmatrix}^\top$, $\Phi(D_x) = \tilde{\Lambda}_1^{21}(D_x) - \Lambda_1^{21}(D_x)\Lambda_1^{11}(D_x)$. Since the operator $\Lambda_1^{21}(D_x)\Lambda_1^{12}(D_x)$ has the LRO, we therefore fulfill the first part of the theorem and can perform similar transformations. Here operators from Definition 3 have the form $\mathcal{A}(D_x) = \operatorname{diag}\{I_r, \; -\Lambda_1^{21}(D_x)\Lambda_1^{12}(D_x)\}$,

$$\Psi_l(D_t, D_x) = \begin{pmatrix} I_r & \\ -\Phi(D_x) & I_{n-r} \end{pmatrix} \mathcal{D}_t \begin{pmatrix} I_r & 0 \\ -\Lambda_1^{21}(D_x) & I_{n-r} \end{pmatrix} \mathcal{D}_t L(x,t).$$

□

Remark 2. The paper [23] graded systems (1), where the operator $\Lambda_1^{22}(D_x)$ has the LRO, as index $(1, l)$. According to this definition, the system from Example 1 has index (1,1) at $\delta \neq 1$ and index (1,2) at $\delta = 1$.

There is also one more important remark to be made. Let the first part of Theorem 2 be satisfied. Then system (12) entails the equality $\Lambda_1^{22}(D_x)z_2 = -\Lambda_1^{21}(D_x)z_1 + g_2$, where $\begin{pmatrix} z_1^\top & z_2^\top \end{pmatrix}^\top = z$.

Using formula (10), express z_1 through z_2 and substitute the result into the first block equations of the DAE (12). We arrive at the system of integral differential equations resolved with respect to the evolutionary term

$$D_t z_1 + [\Lambda_1^{21}(D_x) - \Lambda_1^{12} W(D_x)\Lambda_1^{21}(D_x)]z_1 = h(x,t), \tag{16}$$

where $h(x,t) = g_1 - \Lambda_1^{12}(D_x)X_d(x,t)c(t) - \Lambda_1^{12}(D_x)W(D_x)g_2$. Eq. (16) without its integral part is a linear differential equation, which can be further investigated to find out whether it belongs to the class of hyperbolic, parabolic, or elliptic equations.

A similar system can be derived if the second condition of Theorem 2 is satisfied.

3. Hyperbolicity Criteria for Singular Systems

In this section we discuss techniques for finding the index of system (1) as well as criteria for assigning the system to a certain type. It is quite challenging to actually construct the matrices $L(x,t)$ and $R(x,t)$ that would transform the original system to the form (16) and possess the same smoothness as the matrix $A(x,t)$. For example, consider

$$A(x,t) = \begin{pmatrix} 1-\sin\omega(x,t) & \cos\omega(x,t) \\ \cos\omega(x,t) & 1+\sin\omega(x,t) \end{pmatrix},$$

where $g(x,t)$ is some arbitrary smooth function. It can be readily seen that this matrix has a constant rank in any domain $\mathbf{U}$. Below, we will focus on hyperbolic systems only.

If system (1) is regular, i.e. det $A(x,t) \neq 0 \; \forall (x,t) \in \mathbf{U}$, the hyperbolicity is understood as in [26]. In what follows, we provide criteria which, in terms of input data, guarantee that system (1) has index 1 with respect to t in the domain $\mathbf{U}$ and possesses an implicit hyperbolic structure.

Definition 4. *If the matrix pencil $\lambda \mathrm{A}(x,t) + \mathrm{B}(x,t)$ satisfies the conditions:*

1. $\mathbf{r}[\mathrm{A}(x,t)] = r$; 2. $\det[\lambda\mathrm{A}(x,t) + \mathrm{B}(x,t)] = \mathbf{a}_0(x,t)\lambda^r + \cdots, \; \mathbf{a}_0(x,t) \neq 0 \; \forall (x,t) \in \mathbf{U}$,

then we say that the pencil satisfies the rank-degree criterion in the domain $\mathbf{U}$.

Definition 5. *If the pencil of continuous matrices $\lambda\mathrm{A}(x,t) + \mu\mathrm{B}(x,t) + \mathrm{C}(x,t)$ satisfies the conditions:*

1. $\mathbf{r}[\mathrm{A}(x,t)] = r_1 < n, \; \mathbf{r}[(\mathrm{A}(x,t)|\mathrm{B}(x,t))] = r_1 + r_2 < n$;

2. $\det[\lambda\mathrm{A}(x,t) + \mu\mathrm{B}(x,t) + \mathrm{C}(x,t)] = \mathbf{a}_0(x,t)\lambda^{r_1}\mu^{r_2} + \cdots, \; \mathbf{a}_0(x,t) \neq 0 \; \forall (x,t) \in \mathbf{U}$,

then we say that the pencil satisfies the double rank-degree criterion in the domain $\mathbf{U}$ *(or, in terms of [30], has a simple structure).*

Lemma 3. *If:*

1. $\mathrm{A}(x,t), \; \mathrm{B}(x,t), \; \mathrm{C}(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$;

2. *The matrix pencil* $\lambda \mathrm{A}(x,t) + \mathrm{B}(x,t)$ *satisfies the rank-degree criterion in* $\mathbf{U}$*;*

3. *The matrix pencil* $\lambda \mathrm{A}(x,t) + \mu \mathrm{B}(x,t) + \mathrm{C}(x,t)$ *satisfies the double rank-degree criterion in* $\mathbf{U}$*.*

Then:

1. *Compliance with the the second point of the lemma entails that there exists such square matrices* $P_1(x,t),\ Q_1 x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$ *that* $\det\ P_1(x,t)Q_1(x,t) \neq 0\ \forall (x,t) \in \mathbf{U}$,

$$P_1(x,t)[\lambda \mathrm{A}(x,t) + \mathrm{B}(x,t)]Q_1(x,t) = \lambda \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J(x,t) & 0 \\ 0 & I_l \end{pmatrix}; \tag{17}$$

2. *Compliance with the the third point of the lemma entails that there exists such square matrices* $P(x,t),\ Q(x,t) \in \mathbf{C}^{i,j}(\mathbf{U})$ *that* $\det\ P(x,t)Q(x,t) \neq 0\ \forall (x,t) \in \mathbf{U}$,

$$P(x,t)[\lambda \mathrm{A}(x,t) + \mu \mathrm{B}(x,t) + \mathrm{C}(x,t)]Q(x,t) =$$

$$= \lambda \begin{pmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} J(x,t) & 0 & B_{13}(x,t) \\ 0 & I_\varrho & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} C_{11}(x,t) & C_{12}(x,t) & 0 \\ C_{21}(x,t) & C_{22}(x,t) & 0 \\ 0 & 0 & I_\nu \end{pmatrix}, \tag{18}$$

where $r +_\varrho + \nu = n,\ J(x,t),\ B_{13}(x,t),\ C_{ij}(x,t),\ i = 1,2$ *are the matrix blocks of corresponding dimensions.*

Remark 3. If $\varrho = n - r$ (here this is equivalent to $\mathrm{rank}(\mathrm{A}(x,t)|B(x,t)) = n\ \forall (x,t) \in \mathbf{U}$), then the simple structure condition coincides with the rank-degree criterion. When the matrices of the pencil $\mathrm{A}(x,t) + \mu \mathrm{B}(x,t) + \mathrm{C}(x,t)$ depend only on t and the matrices $\mathrm{A}(x,t),\ (\mathrm{A}(x,t)|B(x,t))$ have a constant rank in the domain, the lemma on reducibility to the form (18) was announced in [31]. The lemma for two variables was proved in [23]. In this work we omit the requirement for the ranks to be constant because they follow from condition 2 of Definition 5.

Theorem 3. *Let in system (1):*

1. $A(x,t),\ B_\rho(x,t),\ \rho = 1,\ C(x,t) \in \mathbf{C}^{i,j}(\mathbf{U}),\ i,j \geq 1;$

2. *The matrix pencil* $\lambda A(x,t) + B_1(x,t)$ *satisfy the rank-degree criterion in* $\mathbf{U}$ *or matrix pencil* $\lambda A(x,t) + \mu B_1(x,t) + C(x,t)$ *have a simple structure in* $\mathbf{U}$*;*

3. *All roots of the polynomial*

$$\det[\lambda A(x,t) + D(x,t)] = 0, \tag{19}$$

where $D(x,t) = B_1(x,t) + [I_n - S(x,t)S^+(x,t)]C(x,t),\ S(x,t) = (A(x,t)\,|\,B_1(x,t))$, *are real and simple, and*

$$\delta < \lambda_1(x,t) < \lambda_2(x,t) < \cdots < \lambda_p(x,t),\ \lambda_{p+1}(x,t) = 0, \cdots, \lambda_r(x,t) = 0\ \forall (x,t) \in \mathbf{U},$$

where δ *is some real number.*

Then:

1. *System (1) has index 1 in the domain* $\mathbf{U}$ *with respect to* t*;*
2. *System (1) is implicitly hyperbolic;*
3. *System (1) has index* $(1,0)$ *if matrix pencil* $\lambda A(x,t)+B_1(x,t)$ *satisfy the rank-degree criterion in* $\mathbf{U}$ *in terms of Remark 2;*
4. *System (1) has index* $(1,0)$ *if matrix pencil* $\lambda A(x,t)+\mu B_1(x,t)+C(x,t)$ *have a simple structure in* $\mathbf{U}$ *in terms of Remark 2.*

Proof. Multiply (1) by the matrix P and introduce the change of variable $u = Qz$, where $P,\ Q$ are matrices from Lemma 3. We obtain

$$\begin{pmatrix} I_r & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} D_t z + \begin{pmatrix} J & 0 & B_{13}\\ 0 & I_\varrho & 0\\ 0 & 0 & 0\end{pmatrix} D_x z + \begin{pmatrix} G_{11} & G_{12} & G_{13}\\ G_{21} & G_{22} & G_{23}\\ 0 & 0 & I_\nu\end{pmatrix} z = Pf, \tag{20}$$

where $G_{ij}(x,t),\ i=1,2,\ j=1,2,3$ are blocks of the matrix $PAD_tQ + PBD_xQ + PCQ$. It is readily seen that if we multiply the last line by $G_{13},\ G_{23}$ and deduct the result from the first and second lines, we can turn the latter ones into zero. Therefore, the matrix P can initially be chosen so that these blocks are zero. Now prove that the eigenvalues of the matrix J coincide with the roots of the polynomial (19). Consider a polynomial

$$\det\ P\det[\lambda A + D]\det\ Q = \det[\lambda PAQ + PBQ + P(I_n - S\tilde{Q}\tilde{Q}^{-1}S^{+})P^{-1}PCQ] =$$

$$= \det\left[\lambda\begin{pmatrix} I_r & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix} J & 0 & B_{13}\\ 0 & I_\varrho & 0\\ 0 & 0 & 0\end{pmatrix} + Z\begin{pmatrix} G_{11} & G_{12} & 0\\ G_{21} & G_{22} & 0\\ 0 & 0 & I_\nu\end{pmatrix}\right],$$

where $\tilde{Q} = \mathrm{diag}\{Q,\ Q\},\ Z = P(I_n - S\tilde{Q}\tilde{Q}^{-1}S^{+})P^{-1}$. Direct calculation shows that $Z = \begin{pmatrix} 0 & Z_1\\ 0 & I_p\end{pmatrix}$, where Z_1 is some block, and $\det\ P\det[\lambda A + D]\det\ Q = \det\ PQ\det[\lambda I_r + J]$. Hence, according to [26], we can choose the matrices $P,\ Q$ so that the matrix J will be diagonal. Rewrite system (20) as

$$\begin{pmatrix} I_p & 0 & 0 & 0\\ 0 & I_q & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} D_t z + \begin{pmatrix} J_1 & 0 & 0 & B_{14}\\ 0 & 0 & 0 & B_{24}\\ 0 & 0 & I_\varrho & 0\\ 0 & 0 & 0 & 0\end{pmatrix} D_x z + \begin{pmatrix} G_{11} & G_{12} & G_{13} & 0\\ G_{21} & G_{22} & G_{23} & 0\\ G_{31} & G_{32} & G_{33} & 0\\ 0 & 0 & 0 & I_\nu\end{pmatrix} z = Pf, \tag{21}$$

where $J_1 = \mathrm{diag}\{\lambda_1, \cdots, \lambda_p\}$. If we write down (21) as the DAE (12), then the corresponding blocks take the form

$$\Lambda_1^{11}(D_x) = \begin{pmatrix} J_1 & 0\\ 0 & 0\end{pmatrix} D_x + \begin{pmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix},\ \Lambda_1^{12}(D_x) = \begin{pmatrix} 0 & 0\\ B_{14} & B_{24}\end{pmatrix} D_x + \begin{pmatrix} G_{13} & 0\\ G_{23} & 0\end{pmatrix},$$

$$\Lambda_1^{21}(D_x) = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix} D_x + \begin{pmatrix} G_{11} & G_{12}\\ 0 & 0\end{pmatrix},\ \Lambda_1^{22}(D_x) = \begin{pmatrix} I_\varrho & 0\\ 0 & 0\end{pmatrix} D_x + \begin{pmatrix} G_{33} & 0\\ 0 & I_\nu\end{pmatrix},$$

where the operator $\Lambda_1^{22}(D_x)$ has an LRO of the form $\mathrm{diag}\{I_\varrho,\ D_xI_\nu\}$. Taking into consideration the form of the matrix J, this fact proves the theorem. □

4. The PDAEs Based Mathematical Models

In view of what has been said above, consider modelling of some processes in power plants. Such models include equations describing fluid motion (for instance, water, oil fuel etc.) in pipelines of the network. The motion of incompressible, viscous liquid substances is described by a system of the Navier – Stokes equations, which can be written down in the form of a PDAE

$$D_t \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} + \begin{pmatrix} -\Delta I_3 & \mathrm{grad} \\ \mathrm{div} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} + \begin{pmatrix} \varpi(\mathrm{u})\mathrm{u} \\ 0 \end{pmatrix} = \begin{pmatrix} g(Z,t) \\ 0 \end{pmatrix}, \qquad (22)$$

where $\mathrm{u} = (u_1\ u_2\ u_3)^\top$, $g(Z,t) = (g_1(Z,t)\ g_2(Z,t)\ g_3(Z,t)\ 0)^\top$ is a given vector-function. $u_j \equiv u_j(Z,t)$, $j = \overline{1,3}$ are coordinate velocities of fluid particles at the point $(Z,t) = (x,y,z,t)$, $\mathrm{p} = \mathrm{p}(Z,t)$ is a pressure at the point (Z,t),

$$\Delta\mathrm{u} = D_x^2 u_1 + D_y^2 u_2 + D_z^2 u_3,\ \mathrm{div}\ \mathrm{u} = D_x u_1 + D_y u_2 + D_z u_3,\ \mathrm{grad}\ \mathrm{u} = (D_x u_1\ D_y u_2\ D_z u_3)^\top$$

are the Laplas operator, divergence, and gradient, correspondignly; $\varpi(\mathrm{u})$ is the Jacobian of the vector-function u. The linearized version of (24) is called the Stokes system

$$\Lambda(D_t, D_x, D_y, D_z)U =$$

$$= D_t \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} + \begin{pmatrix} -\Delta I_3 & \mathrm{grad} \\ \mathrm{div} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} = \begin{pmatrix} g(Z,t) \\ 0 \end{pmatrix},\ U = \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix}. \qquad (23)$$

The system is written in a dimensionless form, i.e. it is assumed that fluid viscosity and density are equal to 1. Various forms of systems (24), (23) have been studied in the immense number of research works. In particular, system (23) was considered on the basis of the transition to (4) (see, for example, [6], [12]). A number of Russian and foreign researchers tried to apply the DAEs theory to the investigation of (23), and the paper [32] seems to be the first work of such kind.

System (23) has the same structure as the DAE (12), so if we set $\Lambda_1^{11} = \Delta I_3$, $\Lambda_1^{12} = -\mathrm{grad}$, $\Lambda_1^{21} = \mathrm{div}$, $\Lambda_1^{22} = 0$, then, by repeating the reasoning of Theorem 2 and taking into consideration $\mathrm{divgrad} = \Delta$, we obtain

$$\Psi_2 \circ \Lambda(D_t, D_x, D_y, D_z)U = \begin{pmatrix} I_3 & 0 \\ 0 & -\Delta \end{pmatrix} D_t \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} + \begin{pmatrix} -\Delta I_3 & \mathrm{grad} \\ \Upsilon\Delta I_3 & -\Upsilon\mathrm{grad} \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix},$$

where

$$\Psi_2 = \begin{pmatrix} I_3 & 0 \\ -\Upsilon & 1 \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & D_t \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ -\mathrm{div} & 1 \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & D_t \end{pmatrix},\ \Upsilon = \mathrm{div}\Delta I_3.$$

Using the expression $\begin{pmatrix} I_3 & 0 \\ 0 & D_t \end{pmatrix} = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} D_t$, write down the operator Ψ_2 as a sum from Definition 3

$$\Psi_2 = \begin{pmatrix} I_3 & 0 \\ -\Upsilon & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mathrm{div} & 0 \end{pmatrix} D_t + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} D_t^2.$$

Therefore, we can assume that system (23) has index 2 with respect to t.

Such a transformation is not possible if the DAE (24) is nonlinear. However, we can apply the operator

$$\tilde{\Psi}_2 = \begin{pmatrix} I_3 & 0 \\ -\mathrm{div} & 1 \end{pmatrix} \left[\begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} D_t \right],$$

to the nonlinear system to reduce its index. As a result we get

$$D_t \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} + \begin{pmatrix} -\Delta I_3 & \mathrm{grad} \\ \Upsilon & -\Delta \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} + \begin{pmatrix} \varpi(\mathrm{u})\mathrm{u} \\ -\mathrm{div}\varpi(\mathrm{u})\mathrm{u} \end{pmatrix} = \begin{pmatrix} g(Z,t) \\ -\mathrm{div} g(Z,t) \end{pmatrix}. \quad (24)$$

It is well-known from the DAE theory, that the index reduction considerably increases computational reliability.

The index is preserved whatever approximations we use. For example, expand the desired function and the known function in a Fourier series with respect to spatial variables

$$U(Z,t) = \sum_{\nu=0}^{\infty} U_\nu(t) e^{-\mathrm{i}(\nu,Z)},\ g(Z,t) = \sum_{\nu=0}^{\infty} G_\nu(t) e^{-\mathrm{i}(\nu,Z)},\ \mathrm{i} = \sqrt{-1},$$

where $\nu = (\nu_1, \nu_2, \nu_3),\ \nu_j,\ j = \overline{1,3}$ are integer numbers,

$$U_\nu(t) = (u_{1\nu}(t)\ u_{2\nu}(t)\ u_{3\nu}(t)\ p_\nu(t))^\top,\ G_\nu(t) = (g_{1\nu}(t)\ g_{2\nu}(t)\ g_{3\nu}(t)\ 0)^\top.$$

Substitute the expansions obtained into (23). We derive an infinite sequence of the DAEs

$$\frac{d}{dt} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U_\nu(t) + \begin{pmatrix} -q_\nu & 0 & 0 & -\mathrm{i}\nu_1 \\ 0 & -q_\nu & 0 & -\mathrm{i}\nu_2 \\ 0 & 0 & -q_\nu & -\mathrm{i}\nu_3 \\ -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 & -\mathrm{i}\nu_3 & 0 \end{pmatrix} U_\nu(t) = G_\nu(t), \quad (25)$$

where $q_\nu = \nu_1^2 + \nu_2^2 + \nu_3^2$. System (25) has index 2 in terms of Definition 2, i.e. there exists the operator $\Omega_2 = L_0 + L_1(d/dt) + L_2(d/dt)^2$, where $L_0,\ L_1,\ L_2 - (4 \times 4)$ are matrices with constant elements. The operator can be constructed following the algorithm from Theorem 2.

The method of lines is another way of approximating partial differential equations. For the sake of simplicity, we consider a two-dimensional Navier – Stokes system in the domain $\mathbf{U} = G \times [t_0, t_1],\ G = [0,1] \times [0,1]$. Introduce on G a uniform grid in the x and y directions with the time step h.

Consider the system

$$\dot{v}_{j,k}(t) - \bar{\Delta} v_{j,k}(t) + \Delta_{\bar{x}} p_{j,k}(t) - [v_{j,k}(t)\Delta_x v_{j,k}(t) + w_{j,k}(t)\Delta_x v_{j,k}(t)] = f_{1,j,k},$$
$$\dot{w}_{j,k}(t) - \bar{\Delta} w_{j,k}(t) + \Delta_{\bar{y}} p_{j,k}(t) - [v_{j,k}(t)\Delta_y w_{j,k}(t) + w_{j,k}(t)\Delta_y w_{j,k}(t)] = f_{2,j,k},$$
$$\Delta_x v_{j,k}(t) + \Delta_y w_{j,k}(t) = 0, \quad (26)$$

where the difference operators have the form:

$$\Delta_x \varsigma_{j,k} = (\varsigma_{j+!,k} - \varsigma_{j,k})/h,\ \Delta_{\bar{x}} \varsigma_{j,k} = (\varsigma_{j,k} - \varsigma_{j-1,k})/h,$$

$$\Delta_y \varsigma_{j,k} = (\varsigma_{j,k+1} - \varsigma_{j,k})/h,\ \Delta_{\bar{y}} \varsigma_{j,k} = (\varsigma_{j,k} - \varsigma_{j,k-1})/h,\ \bar{\Delta} = \Delta_x \Delta_{\bar{x}} + \Delta_y \Delta_{\bar{y}},$$

$\varsigma_{j,k}$ is an arbitrary grid function. The values of the desired functions are assumed to have been derived from the initial boundary conditions, if the index is zero.

Introduce a vector-function

$$\mathcal{U}_N(t) = (\zeta_1(t)\ \zeta_2(t)\ \cdots\ \zeta_{N-1}(t))^\top,\ \mathcal{P}_N(t) = (\eta_1(t)\ \eta_2(t)\ \cdots\ \eta_{N-1}(t))^\top,$$

where $N = 1/h,\ \zeta_j(t) = (v_{j,1}(t)\ w_{j,1}(t)\ v_{j,2}(t)\ w_{j,2}(t)\ \cdots\ v_{j,N-1}(t)\ w_{j,N-1}(t)),\ \eta_j(t) = (p_{j,1}(t)\ p_{j,2}(t)\ \cdots\ p_{j,N-1}(t))$ and rewrite (26) as a DAE

$$\begin{pmatrix} I_{2(N-1)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\mathcal{U}}_N(t) \\ \dot{\mathcal{P}}_N(t) \end{pmatrix} + \begin{pmatrix} \mathcal{D}_N & -\mathcal{M}_N^\top \\ \mathcal{M}_N & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U}_N(t) \\ \mathcal{P}_N(t) \end{pmatrix} + \begin{pmatrix} (\mathcal{S}_N \mathcal{U}_N(t), \mathcal{U}_N(t)) \\ 0 \end{pmatrix} =$$
$$= F_N(t), \tag{27}$$

where $\mathcal{D}_N,\ \mathcal{M}_N,\ \mathcal{S}_N$ are the matrices of the appropriate dimension, $F_N(t)$ is a vector-function composed of the functions $f_{1,j,k},\ f_{2,j,k}$ and components of the initial and boundary conditions.

To control where fluid flows as well as to control fluid pressure, it is common to use hydraulic circuits. The hydraulic circuit graph can be presented by a full $(m \times n)$-matrix $\mathcal{A}$ of nodes and lines that identically describes the structure and the orientation of the circuit: $a_{ij} = 1$, if the line i comes from the node j; $a_{ij} = -1$, if the line i comes into the node j; $a_{ij} = 0$, if the node j does not belong to the line i ($i = \overline{1,\ n},\ j = \overline{1,\ m}$). It is assumed that the first and the second Kirchhoff circuit laws are satisfied: 1) at any node the amount of fluid flowing into the node is equal to the amount of fluid flowing out of that node; 2) the sum of pressure drops in any closed loop is zero. The connection between the flow rate of the line i and the pressures $p_{bx,i}(t),\ p_{bix,i}(t)$ on its ends is expressed as

$$p_{bx,i}(t) - p_{bix,i}(t) + h_i(t) = r_i(t)\dot{\mathbf{x}}_i(t) + s_{0,i}\mathbf{x}_i(t) + s_{1,i}\,|\mathbf{x}_i(t)|\,\mathbf{x}_i(t), \tag{28}$$

where $r_i(t) > 0$ is an inertia parameter of the line, $h_i(t)$ is a hydraulic head, $s_{0,i} > 0$ and $s_{1,i} > 0$ are pipe frictions corresponding to the stream-line and turbulent flows. The relations (28) and the equations following from the Kirchhoff laws can be written in the form of the DAE

$$\begin{pmatrix} R(t) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{X}}(t) \\ \dot{\mathbf{P}}(t) \end{pmatrix} + \begin{pmatrix} S_0 & -\mathcal{A}_1^\top \\ \mathcal{A}_1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X}(t) \\ \mathbf{P}(t) \end{pmatrix} + \begin{pmatrix} S_1\,|\mathbf{X}(t)|\,\mathbf{X}(t) \\ 0 \end{pmatrix} =$$
$$= \begin{pmatrix} H(t) + \mathcal{A}_2^\top P^*(t) \\ Q(t) \end{pmatrix}, \tag{29}$$

where $(\mathcal{A}_1^\top\ \mathcal{A}_2^\top) = \mathcal{A}^\top,\ R = \operatorname{diag}\{r_0(t), r_1(t), \ldots, r_n(t)\},\ S_0 = \operatorname{diag}\{s_{0,1}, s_{0,2}, \ldots, s_{0,n}\}$, $S_0 = \operatorname{diag}\{s_{1,1}, s_{1,2}, \ldots, s_{1,n}\},\ |\mathbf{X}(t)| = \{|\mathbf{x}_1(t)|, |\mathbf{x}_2(t)|, \ldots, |\mathbf{x}_n(t)|\},\ \mathbf{X}(t)$ is an n-dimensional vector-function of the flow rates in pipelines; $\mathbf{P}(t)$ is an m_1-dimensional vector-function of the unknown pressures at nodes; $P^*(t)$ is an m_2-dimensional vector-function of the known pressures; $m_1 + m_2 = m$; $H(t)$ is an n-dimensional vector-function of hydraulic heads; $Q(t)$ is an m_1-dimensional vector-function of inflows; $\operatorname{rank}\mathcal{A}_1 = m_1$. It was previously shown in [33] that if we have a non-linear term in the system, there exists an operator $\Omega_2 = L_0 + L_1(d/dt) + L_2(d/dt)^2$, where $L_0,\ L_1,\ L_2$ are constant matrices of the appropriate dimension, that transforms the DAE (29) to the normal form. Due to the fact that the block structure of (27) is identical to that one of (29), the same technique can be applied to prove existence of such an operator for the DAE (27).

Models of complex power plants are typically described by quasi-linear DAEs, the study of which is quite challenging even when we deal with the simplest models. Consider a quasi-linear PDAE that describes heat exchange in a steam straight-through boiler, which can be primitively represented as a pipe with flowing fluid (water, steam, vapor-water) heated by hot gases and emission from the fuel combustion.

The conservation laws allow us to write down the following PDAE

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} D_t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} \mathbf{x}_{\nu,} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{x}_{\kappa,g} \end{pmatrix} D_x \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} +$$

$$+ \begin{pmatrix} c_1[\mathbf{t}(u_1,p) - u_2] \\ -c_1[\mathbf{t}(u_1,p) - u_2] + c_2 u_2 - c_3 u \\ -c_2 u_2 + c_3 u \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{q}(x,t) \\ 0 \end{pmatrix}, \tag{30}$$

where u_1 is fluid heat content; u_2 is pipe wall temperature; u_3 is gas enthalpy; $\mathbf{t}(u_1, p)$ is fluid temperature; p is fluid pressure; $\mathbf{x}_{\nu,}$, $\mathbf{x}_{\kappa,g}$ are the flow and gas rates in the lines ν, κ; $a_{11}, a_{22}, c_1, c_2, c_2$ are some parameters responsible for the circuit general geometry and properties of hear exchange; $\mathbf{q}(x,t)$ is radiation heat flow. The gas flow rates and pressures are found when solving (29).

Problem (30) can be generalized as follows

$$A(x,t)D_t u + B(x,t)D_x u + \mathrm{C}(u,x,t) = f(x,t), \ (x,t) \in \mathbf{U},$$

where $\mathrm{C}(u,x,t)$ is a given in $\mathbf{R}^n \times \mathbf{U}$ vector-function, and applied to investigation of more relevant models.

Acknowledgements. *This work has been supported by the Russian Foundation for Basic Research, grants No. 15-01-03228, 16-51-540002.*

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Received February 28, 2017

УДК 517.9 **DOI: 10.14529/mmp170201**

О ПОНЯТИИ ИНДЕКСА ДИФФЕРЕНЦИАЛЬНО-АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ, ВОЗНИКАЮЩИХ ПРИ МОДЕЛИРОВАНИИ ПРОЦЕССОВ В ЭНЕРГЕТИЧЕСКИХ УСТАНОВКАХ

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В статье рассматриваются некоторые классы линейных и квазилинейных дифференциально-алгебраических уравнений (ДАУ) в частных производных. Под

ДАУ в частных производных в работе понимаются системы с вырожденными во всей области определения матрицами при старших производных искомой вектор-функции. Они не являются системами типа Коши – Ковалевской, и утверждения о разрешимости в общем случае отсутствуют. Конкретным объектом изучения являются эволюционные системы с одной пространственной переменной. Проведены исследования ДАУ высокого порядка, зависящих от параметра. На этой основе введено понятие индекса ДАУ в частных производных. Рассмотрены постановки начально-краевых задач для ДАУ в частных производных. Полученные результаты применяются для анализа моделей процессов тепломассообмена в энергетических установках.

Ключевые слова: дифференциально-алгебраические уравнения; частные производные; интегро-дифференциальные уравнения; пространство решений; индекс; модели энергетических установок.

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Поступила в редакцию 28 февраля 2017 г.