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APPROXIMATION OF THE SOLUTION SET FOR A SYSTEM OF NONLINEAR INEQUALITIES FOR MODELLING A ONE-DIMENSIONAL CHAOTIC PROCESS

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The paper is focused on the modelling of a one-dimensional chaotic process which dynamics is described by a one-parameter nonlinear map. The problem is to estimate the initial condition and model parameter from measurements corrupted by additive errors. The considered guaranteed (set-membership) approach assumes that the prior information about the unknown variables (initial condition, model parameter and measurement errors) is presented as interval estimates. In this context, the estimation problem can be stated as a problem of solving a system of nonlinear inequalities. Due to the nonlinearity, it is not possible to obtain an exact characterization of the solution set. The developed algorithm computes an outer approximation as a union of non-overlapping boxes.

Keywords: chaotic process; nonlinear modelling; guaranteed approach; interval estimate; outer approximation.

Introduction. Consider the model of a one-dimensional chaotic process

$$x_k = f(x_{k-1}, \lambda), \tag{1}$$

where f is a chaotic map [1]. The problem is to estimate the unknown initial condition x_0 and model parameter λ from measurements

$$y_k = x_k + v_k,\ k = 1, 2, \ldots, N, \tag{2}$$

where v_k are measurement errors. The optimization approach is based on the minimization of a cost function that measures the similarity between the data obtained from the model equation (1) and measurements (2) [2]. The most common technique is the least squares method and its modifications (see review in [3]). The main difficulty of the optimization approach is that the cost function becomes extremely complex and has a large number of local minima [4]. Thus it is necessary to use global optimization algorithms, e.g., particle swarm optimization and differential evolution [5]. One of the promising approaches is preprocessing of measurements to specify the set of possible values of the unknown variables (the search set). It decreases the number of local minima of the cost function.

Guaranteed estimation [6–10] assumes that the prior information about the initial condition x_0, parameter λ and measurement errors v_k is presented as interval estimates:

$$x_0 \in X_0 = [\ \underline{x}_0\ ,\ \overline{x}_0\ ]\,,\ \lambda \in \Lambda_0 = [\ \underline{\lambda}_0\ ,\ \overline{\lambda}_0\ ]\,,\ v_k \in V_k = [\ \underline{v}_k\ ,\ \overline{v}_k\ ]\,. \tag{3}$$

The equations (1), (2) and restrictions (3) lead to the following system of nonlinear inequalities:

$$\begin{cases} y_1 - \overline{v}_1 \le f(x_0, \lambda) \le y_1 - \underline{v}_1, \\ y_2 - \overline{v}_2 \le f^2(x_0, \lambda) \le y_2 - \underline{v}_2, \\ \ldots \\ y_N - \overline{v}_N \le f^N(x_0, \lambda) \le y_N - \underline{v}_N, \end{cases} \tag{4}$$

where f^k is the k-fold composition of the map f with itself:

$$f^k(x_0,\lambda) = \underbrace{f(f(f\ldots f}_{k}(x_0,\lambda)\ldots)).$$

Let $P^* \subseteq X_0 \times \Lambda_0$ be the set of all solutions (x_0,λ) of the system (4). Due to the nonlinearity, it is not possible to obtain an exact characterization of the solution set P^*. In this paper, the guaranteed algorithm (GA) [11] is applied to construct an outer approximation of the solution set P^* [7–10]. The GA computes interval estimates X_k, Λ_k that contain the true values of the state x_k and parameter λ, respectively:

$$x_k \in X_k = [\ \underline{x}_k\ ,\ \overline{x}_k\ ]\,,\ \lambda \in \Lambda_k = [\ \underline{\lambda}_k\ ,\ \overline{\lambda}_k\ ]\,.$$

1. Guaranteed Algorithm. The GA is based on interval analysis [7] and the map f is considered as an interval function. In this section, the following notations will be used:

$$\begin{aligned} f(X,\Lambda) &= \{u \mid u = f(x,\lambda), x \in X, \lambda \in \Lambda\}\,,\\ f(x,\Lambda) &= \{u \mid u = f(x,\lambda), \lambda \in \Lambda\}\,,\\ f(X,\lambda) &= \{u \mid u = f(x,\lambda), x \in X\}\,. \end{aligned}$$

The GA is a recurrent procedure that can be used in the forward and backward time directions (denoted by the superscripts "+" and "−", respectively).

Forward GA recursions. Suppose that X_0^+, Λ_0^+ are the prior interval estimates for $k=0$. The following steps represent the computation of the interval estimates X_k^+, Λ_k^+ for $k=1,2,\ldots,N$.

Step 1. The predicted state set $X_{k/k-1}$ is defined by the interval estimates X_{k-1}^+, Λ_{k-1}^+ found at the previous time step:

$$X_{k/k-1} = f(X_{k-1}^+, \Lambda_{k-1}^+).$$

Step 2. The consistent state set Y_k is defined by the observation y_k and the interval estimate V_k of the measurement error v_k:

$$Y_k = \{x \mid x = y_k - v, v \in V_k\} = [\ y_k - \overline{v}_k\ ,\ y_k - \underline{v}_k\ ]\,.$$

Step 3. The interval estimate X_k^+ of the state x_k is the intersection of the predicted state set $X_{k/k-1}$ and the consistent state set Y_k:

$$X_k^+ = X_{k/k-1} \cap Y_k.$$

Step 4. The interval estimate $\Lambda_k^+ \subseteq \Lambda_{k-1}^+$ of the parameter λ is given by

$$\Lambda_k^+ = \left\{\lambda \in \Lambda_{k-1}^+ \mid f(\lambda, X_{k-1}^+) \cap X_k^+ \neq \oslash\right\}.$$

Backward GA recursions. For the last time step, let $X_N^- = X_N^+$, $\Lambda_N^- = \Lambda_N^+$. For $k = N-1, N-2, \ldots, 0$ the interval estimates X_k^-, Λ_k^- are defined by the following steps.

Step 1. The interval estimate $X_k^- \subseteq X_k^+$ of the state x_k is given by

$$X_k^- = \left\{x \in X_k^+ \mid f(x, \Lambda_{k+1}^-) \cap X_{k+1}^- \neq \oslash\right\}.$$

Step 2. The interval estimate $\Lambda_k^- \subseteq \Lambda_{k+1}^-$ of the parameter λ is given by

$$\Lambda_k^- = \left\{\lambda \in \Lambda_{k+1}^- \mid f(\lambda, X_k^-) \cap X_{k+1}^- \neq \oslash\right\}.$$

If the prior information (3) is correct, the result of the forward and backward computations is guaranteed: at every time step k found interval estimates always contain the true values of the unknown variables. In this way, the GA can be used to specify the prior interval estimates X_0^+, Λ_0^+ for the initial condition x_0 and parameter λ: $X_0^- \subseteq X_0^+$, $\Lambda_0^- \subseteq \Lambda_0^+$. In the case of incorrect prior information ($x_0 \notin X_0^+$ or $\lambda \notin \Lambda_0^+$), the result is the empty sets: $X_0^- = \oslash$, $\Lambda_0^- = \oslash$. Thus, the GA is a procedure to verify if the box $P^+ = X_0^+ \times \Lambda_0^+$ contains any solutions of the system (4) and to compute a more accurate outer approximation $P^- = X_0^- \times \Lambda_0^-$ such that $P^* \subseteq P^- \subseteq P^+$.

2. Numerical Example. Consider the chaotic process x_k given by the logistic map

$$x_k = \lambda x_{k-1}(1 - x_{k-1})$$

with the initial condition $x_0 = 0,3$ and parameter $\lambda = 3,7$. The measurement errors v_k are pseudo-random numbers with normal distribution, zero mean and standard deviation $\sigma = 0,1$ (the values are generated by the function *randn* of *Matlab*). The available $N = 30$ measurements y_k are shown in Fig. 1. The prior interval estimates are taken as follows:

$$X_0 = [0,\ 0,5]\,,\ \Lambda_0 = [3,\ 4]\,,\ V_k = [-3\sigma,\ 3\sigma]\,.$$

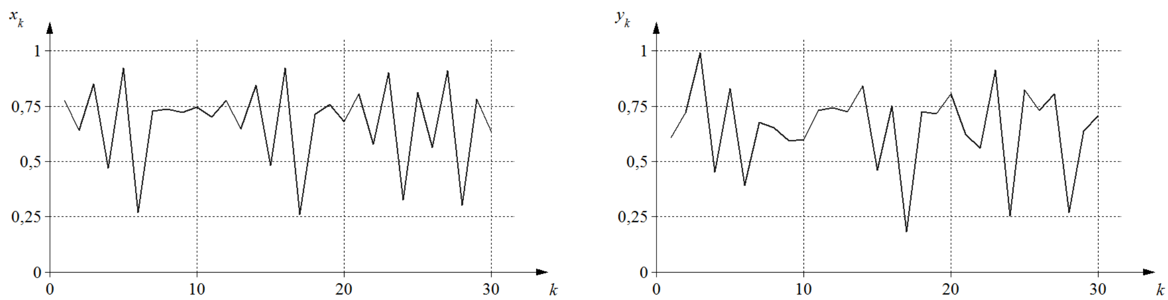

Fig. 1. Chaotic process x_k given by the logistic map and its noisy measurements y_k

The goal is to construct an outer approximation of the solution set P^* as a union of non-overlapping boxes [7, 9, 10]. Let the set $P = X_0 \times \Lambda_0$ be the initial approximation. Obviously the set P can be divided into non-overlapping boxes:

$$P = \bigcup_{i \in I} P^{(i)},\ P^{(i)} = X_0^{(i)} \times \Lambda_0^{(i)}.$$

As mentioned on the previous section, the GA allows to verify if the box $P^{(i)}$ contains any solutions of the system (4) and to compute the box $\widetilde{P}^{(i)} = \widetilde{X}_0^{(i)} \times \widetilde{\Lambda}_0^{(i)}$ such that $\widetilde{P}^{(i)} \subseteq P^{(i)}$, $\widetilde{X}_0^{(i)} \subseteq X_0^{(i)}$, $\widetilde{\Lambda}_0^{(i)} \subseteq \Lambda_0^{(i)}$. As a result, the current approximation can be specified. New outer approximation is the union

$$\widetilde{P} = \bigcup_{i \in \widetilde{I}} \widetilde{P}^{(i)},$$

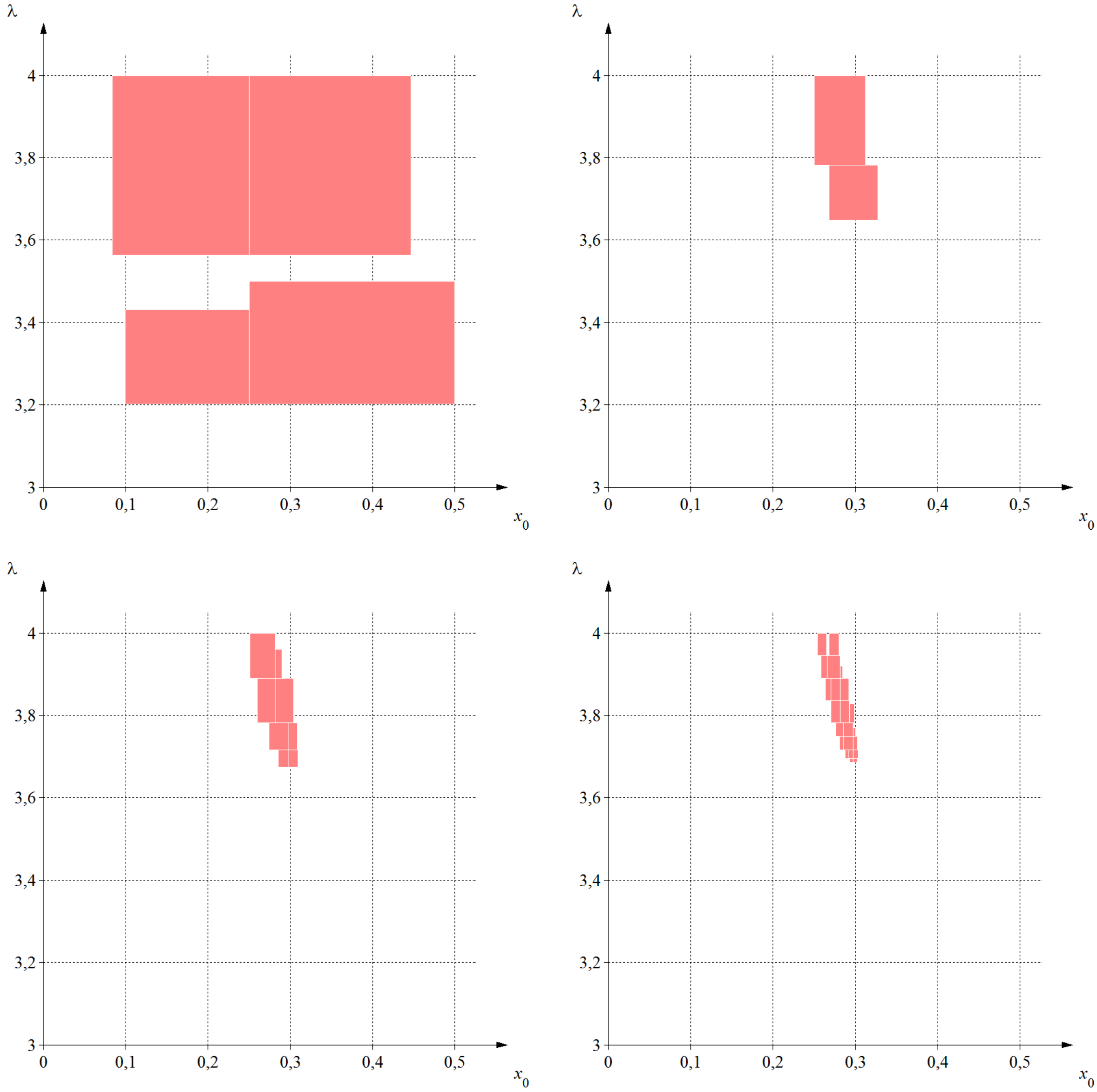

Fig. 2. Evolution of the outer approximation

where $\widetilde{I} \subseteq I$. Then the same procedure can be recursively applied to the boxes $\widetilde{P}^{(i)}$, $i \in \widetilde{I}$. In the considered example, each box is divided into four equal ones. The evolution of the outer approximation is shown in Fig. 2.

Conclusion. Guaranteed approach for modelling a one-dimensional chaotic process leads to solving a system of nonlinear inequalities. The developed algorithm can be applied to construct an outer approximation of the solution set. If the parameter estimation problem is solved by the least squares method, obtained approximation can be used as a set of possible values of the unknown variables. It decreases the number of local minima of the cost function.

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АППРОКСИМАЦИЯ МНОЖЕСТВА РЕШЕНИЙ СИСТЕМЫ НЕЛИНЕЙНЫХ НЕРАВЕНСТВ ПРИ МОДЕЛИРОВАНИИ ОДНОМЕРНОГО ХАОТИЧЕСКОГО ПРОЦЕССА

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В работе рассматривается класс моделей одномерных хаотических процессов, заданных в виде однопараметрических нелинейных отображений. Решается задача оценивания начального условия и параметра модели по зашумленным измерениям.

Применение гарантированного (теоретико-множественного) подхода предполагает, что априорная информация о неизвестных переменных (начальном условии, параметре модели и аддитивных ошибках измерений) представлена в виде интервальных оценок. При данных предположениях задачу оценивания можно интерпретировать как задачу поиска решений системы нелинейных неравенств. При этом вследствие нелинейности точное описание множества решений системы невозможно. Результатом разрабатываемого алгоритма является внешняя аппроксимация множества решений в виде объединения непересекающихся подмножеств.

Ключевые слова: хаотический процесс; нелинейная модель; гарантированный подход; интервальная оценка; внешняя аппроксимация.

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