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NEW APPROXIMATE METHOD FOR SOLVING THE STOKES PROBLEM IN A DOMAIN WITH CORNER SINGULARITY

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In this paper we introduce the notion of an R_ν-generalized solution to the Stokes problem with singularity in a two-dimensional non-convex polygonal domain with one reentrant corner on its boundary in special weight sets. We construct a new approximate solution of the problem produced by weighted finite element method. An iterative process for solving the resulting system of linear algebraic equations with a block preconditioning of its matrix is proposed on the basis of the incomplete Uzawa algorithm and the generalized minimal residual method. Results of numerical experiments have shown that the convergence rate of the approximate R_ν-generalized solution to an exact one is independent of the size of the reentrant corner on the boundary of the domain and equals to the first degree of the grid size h in the norm of the weight space $W^1_{2,\nu}(\Omega)$ for the velocity field components in contrast to the approximate solution produced by classical finite element or finite difference schemes convergence to a generalized one no faster than at an $\mathcal{O}(h^\alpha)$ rate in the norm of the space $W^1_2(\Omega)$ for the velocity field components, where $\alpha < 1$ and α depends on the size of the reentrant corner.

Keywords: corner singularity; weighted finite element method; preconditioning.

Introduction

The weak solution of Maxwell equations considered in a $2D$ polygonal domain with reentrant corner on the boundary does not belong to the Sobolev space $W^1_2(\Omega)$. Such a problem is called a boundary value problem with strong singularity. For the Lamé system, for a example, in a domain with a reentrant corner it is possible to define a weak solution in the space $W^1_2(\Omega)$, but it does not belong to the space $W^2_2(\Omega)$. Such problem is called a problem with weak singularity.

According to the principle of coordinated estimates, the approximate solution to these problems by the classical finite difference and finite element methods converge to the exact one with a rate substantially smaller than one. In [1,2] it was proposed to define the solution of elliptic boundary value problems and Maxwell equations with strong singularity as an R_ν-generalized one. Such a new conception of solution allows to construct weighted finite element methods with first-order convergence rate estimate of the approximate solution to the R_ν-generalized one in the norms of the weighted Sobolev spaces.

In this paper we present our method for solving the Stokes problem. It is well known that the efficient numerical solution of problems in fluid mechanics is of significant engineering interest. There are basically three reasons why the finite element discretization of such problem turns out to be difficult.

Firstly, in the presence of reentrant corner ω, $\omega \in (\pi, 2\pi)$, on the boundary of the domain the solution of the problem is singular even though the input data are sufficiently smooth. The two-dimensional flow of a viscous fluid near the corner was first examined in [3]. It is well known that the generalized solution of the Stokes problem: the velocity components and pressure in a two-dimensional domain Ω with a boundary containing a reentrant angle does not belong to $W_2^2(\Omega)$ and $W_2^1(\Omega)$ respectively (see e.g. [4]). Therefore, the approximate solution produced by standard finite element or finite difference schemes converges to a generalized solution no faster that at an $\mathcal{O}(h^\alpha)$ rate in the norm of the space $W_2^1(\Omega)$, where $\alpha < 1$ depends on the size of the reentrant corner ω for the velocity components (see [5]). In this case the so-called pollution effect can be observed in standard Sobolev and even in weighted Sobolev norms [6]. More recent results on the regularity theory and finite element approximations on domains with reentrant corners can be found in [7] and the references therein. By using special methods for extracting the singular part of the solution near corner points and applying grids refined towards the singularity point, it is possible to construct first-order accurate finite element schemes (see e.g. [8]).

Secondly, the design of LBB-stable method for a velocity and pressure spaces pairs [9].

Thirdly, the spaces enforce mass conservation strongly. Satisfying this criterion leads to more physically relevant solutions, decouples the pressure error from the velocity error, and removes possible instabilities that can arise from poor discrete mass conservation [10]. The specific element pair to achieve pointwise mass conservation of the discrete solution is the Scott–Vogelius element pair [11].

In the present paper we construct the weighted finite element method (see [1,2,12–15]) based on the conception of an R_ν-generalized solution [16–19] of the Stokes problem with a singularity due to a reentrant corner of ω on the boundary of the domain and Scott-Vogelius element pair. Numerical experiments of the model problems have shown that the approximate R_ν-generalized solution produced by weighted finite element method converges to the exact one (velocity) with the rate $\mathcal{O}(h)$ in $\mathbf{W}^1_{2,\nu}(\Omega,\delta)$ norm for all considered sizes of the reentrant corner ω in contrast to the approximate solution produced by classical finite element or finite difference schemes convergence to a generalized one no faster than at an $\mathcal{O}(h^\alpha)$ $(\alpha = \alpha(\omega) < 1)$ rate. The simplicity of the solution determination is an additional benefit of the method for the numerical experiments.

The structure of the paper is as follows. In Section 1 define the R_ν-generalized solution of the Stokes problem with corner singularity. In Section 2 describe the proposed weighted finite element method. In Section 3 construct an iterative process with a block preconditioning matrix. In Section 4 present the results of numerical experiments. Finally, some concluding remarks are given in Section 5.

1. R_ν-Generalized Solution

Let $\mathbf{R}^2$ denote the two-dimensional Euclidean space, $\mathbf{x} = (x_1, x_2)$ be its arbitrary element, $\|\mathbf{x}\| = \left(x_1^2 + x_2^2\right)^{1/2}$ and $d\mathbf{x} = dx_1\, dx_2$. Let $\Omega \subset \mathbf{R}^2$ be a bounded non-convex polygonal domain with a boundary Γ containing a reentrant angle with its vertex placed at the origin, and let $\bar{\Omega}$ be a closure of Ω, i.e. $\bar{\Omega} = \Omega \cup \Gamma$. Denote by $\Omega'_\delta = \{\mathbf{x} \in \bar{\Omega} : \|\mathbf{x}\| \leq \delta < 1, \delta > 0\}$ the part of a δ-neighbourhood of the point $(0,0)$ that lies in $\bar{\Omega}$. Define a weight function $\rho(\mathbf{x})$, such that $\rho(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|, \mathbf{x} \in \Omega'_\delta, \\ \delta \quad, \mathbf{x} \in \bar{\Omega} \setminus \Omega'_\delta. \end{cases}$

Let $L_{2,\beta}(\Omega)$ denote the weighted space of functions with bounded norm

$$\|v\|_{L_{2,\beta}(\Omega)} = \Big(\int\limits_{\Omega} \rho^{2\beta}(\mathbf{x}) v^2(\mathbf{x}) d\mathbf{x}\Big)^{1/2}$$

and $W^1_{2,\beta}(\Omega)$ denote the weighted space of functions with bounded norm

$$\|v\|_{W^1_{2,\beta}(\Omega)} = \Big(\sum_{|m|\le 1} \|\rho^{\beta}(\mathbf{x})|D^m v(\mathbf{x})|\|^2_{L_2(\Omega)}\Big)^{1/2}, \qquad (1)$$

where $D^m v(\mathbf{x}) = \frac{\partial^{|m|} v}{\partial x_1^{m_1} \partial x_2^{m_2}}$, $|m| = m_1 + m_2, m_i \ge 0, i = 1,2$ – integer. For vector functions $\mathbf{v} = (v_1, v_2)$ we define weighted spaces $\mathbf{L}_{2,\beta}(\Omega)$ and $\mathbf{W}^1_{2,\beta}(\Omega)$ with norms $\|\mathbf{v}\|_{\mathbf{L}_{2,\beta}(\Omega)} = \Big(\|v_1\|^2_{L_{2,\beta}(\Omega)} + \|v_2\|^2_{L_{2,\beta}(\Omega)}\Big)^{1/2}$ and $\|\mathbf{v}\|_{\mathbf{W}^1_{2,\beta}(\Omega)} = \Big(\|v_1\|^2_{W^1_{2,\beta}(\Omega)} + \|v_2\|^2_{W^1_{2,\beta}(\Omega)}\Big)^{1/2}$ respectively.

Let $W^1_{2,\beta}(\Omega, \delta)$, for $\beta > 0$, denote a set of functions $v(\mathbf{x})$ from the space $W^1_{2,\beta}(\Omega)$ satisfying the following conditions:

$$\int\limits_{\bar\Omega \backslash \Omega'_\delta} \rho^{2\beta}(\mathbf{x}) v^2(\mathbf{x}) d\mathbf{x} \ge C_1 > 0, \qquad |D^m v(\mathbf{x})| \le C_2 \Big(\frac{\delta}{\rho(\mathbf{x})}\Big)^{\beta+m} \quad \mathbf{x} \in \Omega'_\delta, \qquad (2)$$

where $m = 0, 1$ and $C_2 > 0$ be a constant independent of m, with a norm (1). Let $L_{2,\beta}(\Omega, \delta)$ be a set of functions from the space $L_{2,\beta}(\Omega)$, which satisfy the conditions (2) (for $m = 0$) with a norm of the space $L_{2,\beta}(\Omega)$. Also, $L^0_{2,\beta}(\Omega, \delta) = \{q \in L_{2,\beta}(\Omega, \delta) : \int\limits_{\Omega} \rho^{\beta}\, q d\mathbf{x} = 0\}$.

The set $\overset{o}{W}{}^1_{2,\beta}(\Omega, \delta)(\overset{o}{W}{}^1_{2,\beta}(\Omega, \delta) \subset W^1_{2,\beta}(\Omega, \delta))$ is defined as a closure in norm (1) of the set of infinitely differentiable compactly supported functions in Ω, that satisfy conditions (2). We say that $\varphi(\mathbf{x}) \in W^{1/2}_{2,\beta}(\Gamma, \delta)$, if there exists a function $\Phi(\mathbf{x})$ from $W^1_{2,\beta}(\Omega, \delta)$ such that $\Phi(\mathbf{x})|_\Gamma = \varphi(\mathbf{x})$ and $\|\varphi\|_{W^{1/2}_{2,\beta}(\Gamma,\delta)} = \inf\limits_{\Phi|_\Gamma = \varphi} \|\Phi\|_{W^1_{2,\beta}(\Omega,\delta)}$.

For vector functions $\mathbf{v} = (v_1, v_2)$ define sets $\mathbf{L}_{2,\beta}(\Omega, \delta)$ and $\mathbf{W}^1_{2,\beta}(\Omega, \delta)$ such that $v_i \in L_{2,\beta}(\Omega, \delta)$ and $v_i \in W^1_{2,\beta}(\Omega, \delta)$ with a bounded norm of spaces $\mathbf{L}_{2,\beta}(\Omega)$ and $\mathbf{W}^1_{2,\beta}(\Omega)$ respectively. Similarly for vector functions we define sets $\overset{o}{\mathbf{W}}{}^1_{2,\beta}(\Omega, \delta)$ and $\mathbf{W}^{1/2}_{\beta}(\Gamma, \delta)$.

The Stokes problem is to find the velocity field $\mathbf{u} = (u_1, u_2)$ and pressure p which satisfy the system of differential equations and the boundary conditions

$$-\bar\nu \triangle \mathbf{u} + \nabla p = \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = 0, \qquad \text{in} \quad \Omega, \qquad (3)$$

$$\mathbf{u} = \mathbf{g}, \qquad \text{on} \quad \Gamma. \qquad (4)$$

Here $\mathbf{f} = (f_1, f_2)$ and $\mathbf{g} = (g_1, g_2)$ are defined in Ω and on Γ respectively, $\bar\nu$ is the kinematic viscosity which is related to the Reynolds number Re of the flow by $\bar\nu = \frac{1}{Re}$.

Define the bilinear and linear forms

$$a(\mathbf{u}, \mathbf{v}) = \int\limits_{\Omega} \bar\nu\, \nabla \mathbf{u} \cdot \nabla(\rho^{2\nu} \mathbf{v}) d\mathbf{x}, \qquad b(\mathbf{v}, p) = -\int\limits_{\Omega} p\, \operatorname{div}\, (\rho^{2\nu} \mathbf{v}) d\mathbf{x},$$

$$c(\mathbf{u}, q) = -\int\limits_{\Omega} \rho^{2\nu}\, q\, \operatorname{div}\, \mathbf{u}\, d\mathbf{x}, \qquad l(\mathbf{v}) = \int\limits_{\Omega} \rho^{2\nu}\, \mathbf{f} \cdot \mathbf{v} d\mathbf{x}.$$

Definition 1. *The pair of functions* $(\mathbf{u}_\nu(\mathbf{x}), p_\nu(\mathbf{x})) \in \mathbf{W}^1_{2,\nu}(\Omega,\delta) \times L^0_{2,\nu}(\Omega,\delta)$ *is called an* R_ν*-generalized solution of the Stokes problem (3), (4), if* $\mathbf{u}_\nu(\mathbf{x})$ *satisfies the boundary condition (4) almost everywhere on* Γ *and integral identities*

$$a(\mathbf{u}_\nu, \mathbf{v}) + b(\mathbf{v}, p_\nu) = l(\mathbf{v}),$$
$$c(\mathbf{u}_\nu, q) = 0$$

hold for any pair $(\mathbf{v}(\mathbf{x}), q(\mathbf{x})) \in \overset{o}{\mathbf{W}}{}^1_{2,\nu}(\Omega,\delta) \times L^0_{2,\nu}(\Omega,\delta)$*, where the right-hand side functions* $\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega,\delta)$, $\mathbf{g} \in \mathbf{W}^{1/2}_{2,\beta}(\Gamma,\delta)$, $\nu \geq \beta$.

2. The Weighted Finite Element Scheme

The weighted finite element scheme for the Stokes problem (3), (4) is constructed relying on the definition of an R_ν-generalized solution. For this purpose, we construct the triangulation Υ_h which is a barycenter refinement of a quasi-uniform triangulation T_h of $\overline{\Omega}$ [20]. The domain $\overline{\Omega}$ is divided into a finite number of triangles L_i, $L_i \in T_h$ (macro-element). Each L_i is refined as stated above into three triangles K_{i_j} (finite element), $K_{i_j} \in \Upsilon_h$ (barycenter refinement) with vertices R_l and midpoints S_k. Then,
1)$R^{vel} = R^{vel}_\Omega \cup R^{vel}_\Gamma = \{R_l \cup S_k\}$, where R^{vel}_Ω and R^{vel}_Γ are sets of triangulation nodes for the velocity components in Ω and on Γ respectively;
2)$R^{pres} = \{Q_l\}$ is the set of triangulation nodes for the pressure, where a node Q_l coincides with a node R_k on the corresponding K_{i_j}.

Denote by $\Omega_h = \bigcup\limits_{K_s \in \Upsilon_h} K_s$ the union of all finite elements with sides of order h. Now we introduce Scott–Vogelius (SV) element pair [11] (case $k = 2$). In short, polynomials of degree *two* and *one* are used to approximate the velocity components and pressure, spaces X^h and Z^h respectively:
$X^h = \{v^h \in C(\Omega) : v^h|_K \in P_2(K), \forall K \in \Upsilon_h\}(\mathbf{X}^h = X^h \times X^h)$;
$Z^h = \{z^h \in L_2(\Omega) : z^h|_K \in P_1(K), \forall K \in \Upsilon_h, \int\limits_\Omega z^h d\mathbf{x} = 0\}$.

The SV element is very interesting from the mass conservation point of view since its discrete velocity space $\mathbf{X}^h$ and its discrete pressure space Z^h satisfy an important property, namely $\operatorname{div} \mathbf{X}^h \subset Z^h$. Thus, using SV elements, weak mass conservation via $\int\limits_\Omega \operatorname{div} \mathbf{w}^h z^h d\mathbf{x} = 0 \ \forall z^h \in Z^h$ implies strong (pointwise) mass conservation. We can choose the special test function $z^h = \operatorname{div} \mathbf{w}^h$ to get $\| \operatorname{div} \mathbf{w}^h\|_{L_2(\Omega)} = 0$. In [21] it was shown that the SV space pair ($k = 2$) is LBB-stable.

Now we introduce a special set of basis functions and construct a weighted finite element scheme for the Stokes problem (3), (4). Each node $M_k \in R^{vel}_\Omega (N_l \in R^{pres})$ is associated with a function

$$\theta_k(\mathbf{x}) = \rho^{\nu^*}(\mathbf{x}) \cdot \varphi_k(\mathbf{x}), \ \Big(\chi_l(\mathbf{x}) = \rho^{\mu^*}(\mathbf{x}) \cdot \psi_l(\mathbf{x})\Big), k = 0,1,\ldots, (\ l = 0,1,\ldots),$$

where $\varphi_k \in X^h, \varphi_k(M_j) = \delta_{kj}$ for $k,j = 0,1,\ldots \Big(\psi_l \in Z^h, \psi_l(N_j) = \delta_{lj},\ l,j = 0,1,\ldots\Big)$; δ_{ms} is Kronecker delta, ν^* and μ^* are real constants.

The sets V^h and Q^h for the velocity components and pressure are defined as the linear span of the system of basis functions $\{\theta_k\}_k$ and $\{\chi_l\}_l$ respectively. In V^h we consider the subset $V_0^h = \{v^h \in V^h : v^h(M_k)|_{M_k \in R_\Gamma^{vel}} = 0\}$. Associated with the constructed triangulation, the finite element approximation of the displacement velocity components and pressure have the form

$$u_{\nu,1}^h(\mathbf{x}) = \sum_k d_{2k}\,\theta_k(\mathbf{x}),\ u_{\nu,2}^h(\mathbf{x}) = \sum_k d_{2k+1}\,\theta_k(\mathbf{x}),\ p_\nu^h(\mathbf{x}) = \sum_l e_l\,\chi_l(\mathbf{x}), \tag{5}$$

where $d_j = \rho^{-\nu^*}(M_{[j/2]})\,\tilde{d}_j, e_i = \rho^{-\mu^*}(N_i)\,\tilde{e}_i$. The coefficients d_j and e_i in (5) are defined from the system of equations (see (8)) below.

The corresponding velocity field sets are denoted by $\mathbf{V}^h = V^h \times V^h$ and $\mathbf{V}_0^h = V_0^h \times V_0^h$. Obviously, $\mathbf{V}^h \subset \mathbf{W}_{2,\nu}^1(\Omega_h,\delta), \mathbf{V}_0^h \subset \overset{o}{\mathbf{W}}{}_{2,\nu}^1(\Omega_h,\delta)$ and $Q^h \subset L_{2,\nu}^0(\Omega_h,\delta)$.

Definition 2. *The approximate R_ν-generalized solution of the Stokes problem (3), (4) produced by the weighted finite element method is the pair $(\mathbf{u}_\nu^h(\mathbf{x}), p_\nu^h(\mathbf{x})) \in \mathbf{V}^h \times Q^h$ such that each component of $\mathbf{u}_\nu^h(\mathbf{x})$ at nodes of the set R_Γ^{vel} satisfies the boundary condition (4) and, for an arbitrary pair $(\mathbf{v}^h(\mathbf{x}), p^h(\mathbf{x})) \in \mathbf{V}_0^h \times Q^h$ and $\nu \geq \beta$, we have the equalities*

$$a(\mathbf{u}_\nu^h, \mathbf{v}^h) + b(\mathbf{v}^h, p_\nu^h) = l(\mathbf{v}^h), \tag{6}$$

$$c(\mathbf{u}_\nu^h, q^h) = 0, \tag{7}$$

where $\mathbf{u}_\nu^h = (u_{\nu,1}^h, u_{\nu,2}^h)$ and $\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega,\delta)$, $\mathbf{g} \in \mathbf{W}_{2,\beta}^{1/2}(\Gamma,\delta)$.

The finite element problem (6), (7) generates a system of linear equations with a saddle point matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^T & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix}. \tag{8}$$

In our case, $\mathbf{A}$ is a positive definite square matrix, $\mathbf{B}$ and $\mathbf{C}^T$ are non-square matrices, $\zeta = \mathbf{u}_\nu^h, \eta = p_\nu^h, \mathbf{y} = \mathbf{F}^h, \mathbf{z} = \mathbf{0}$.

3. Iterative Method

The system of linear algebraic equations (8) is large and sparse, making direct solutions infeasible. We construct a convergent iterative process [22] of the following form:
1) select an initial guess η^0, ζ^0 to the solution of (8);
2) for $k = 0, 1, 2, \ldots,$ until converge do;
3) compute $\zeta^{k+1} = \zeta^k + \hat{\mathbf{A}}^{-1}(\mathbf{y} - \mathbf{A}\zeta^k - \mathbf{B}\eta^k)$;
4) compute $\eta^{k+1} = \eta^k + \hat{\mathbf{S}}^{-1}(\mathbf{C}^T\zeta^{k+1} - \mathbf{z})$;
5) end do,
where $\hat{\mathbf{A}}$ and $\hat{\mathbf{S}}$ are preconditioning matrices to $\mathbf{A}$ and the Schur complement $\mathbf{S} = \mathbf{C}^T\mathbf{A}^{-1}\mathbf{B}$ respectively.

To construct matrix $\hat{\mathbf{A}}$, we use an incomplete $\mathbf{LU}$ factorization of $\mathbf{A}$ – $\mathbf{ILU}(0)$ [23], i.e., $\hat{\mathbf{A}} = \hat{\mathbf{L}} \cdot \hat{\mathbf{U}}$, where $\hat{\mathbf{L}}$ and $\hat{\mathbf{U}}$ are lower and upper triangular matrices respectively. At each iteration in 3, we solve the problem $\mathbf{Aq} = \chi$ with the left preconditioner $\hat{\mathbf{A}}$. We use the generalized minimal residual method (GMRES(n)) [23]. The method approximates the solution by the vector in a n-th Krylov subspace with minimal residual. Let $\mathbf{r}_0 = \hat{\mathbf{A}}^{-1}(\chi - \mathbf{Aq})$, then the Arnoldi loop constructs an orthogonal basis of the left preconditioned n-th Krylov subspace: $\mathrm{Span}\{\mathbf{r}_0, \hat{\mathbf{A}}^{-1}\mathbf{A}\mathbf{r}_0, \ldots, (\hat{\mathbf{A}}^{-1}\mathbf{A})^{n-1}\mathbf{r}_0\}, n = 10$.

Further, we construct an auxiliary matrix $\tilde{\mathbf{S}}$ to $\hat{\mathbf{S}}$, as a weighted mass matrix $\mathbf{M}_p^{\nu,\mu^*}$ of the pressure space, on each $L \in \Upsilon_h$:

$$(\mathbf{M}_p^{\nu,\mu^*})_{lj} = \frac{1}{\bar{\nu}} \int\limits_L \rho^{2(\nu+\mu^*)}\, \psi_l(\mathbf{x})\, \psi_j(\mathbf{x}) d\mathbf{x},\ \ l,j = 0,1,\dots .$$

Then, we define the diagonal matrix $\bar{\mathbf{S}} = \bar{\mathbf{M}}_p^{\nu,\mu^*}$, where $\left(\bar{\mathbf{M}}_p^{\nu,\mu^*}\right)_{ii} = \sum\limits_k \left(\mathbf{M}_p^{\nu,\mu^*}\right)_{ik}$. It is well known (see [24]), that such diagonal lumping is a good preconditioner to the initial weighted mass matrix.

It derives from the above that at each iteration in 4 one finds a vector $\psi^\diamond := \hat{\mathbf{S}}^{-1}\theta$ as a solution of internal procedure:
1) $\phi_0 = \mathbf{0}$; 2) $\phi_m = \phi_{m-1} + \bar{\mathbf{S}}^{-1}(\theta - \tilde{\mathbf{S}}\phi_{m-1})\,(m = 1,\dots,M)$; 3) $\psi^\diamond = \phi_M$.
We use restart GMRES(k): (Span$\{\bar{\mathbf{r}}, \bar{\mathbf{S}}^{-1}\tilde{\mathbf{S}}\bar{\mathbf{r}}, \dots, (\bar{\mathbf{S}}^{-1}\tilde{\mathbf{S}})^{k-1}\,\bar{\mathbf{r}}\}$, $\bar{\mathbf{r}} = \bar{\mathbf{S}}^{-1}(\theta - \tilde{\mathbf{S}}\phi_{m-1}), k = 5$).

4. Numerical Experiments

In this section we present the results of numerical experiments to illustrate the behaviour of our method applied to the Stokes problem (3), (4) with viscosity $\bar{\nu} = 1$.

Let $\Omega_i = (-1,1) \times (-1,1) \setminus \bar{D}_i$ are non-convex polygonal domains with one reentrant corner ω_i : $\omega_1 = \frac{3\pi}{2}, \omega_2 = \frac{5\pi}{4}, \omega_3 = \frac{9\pi}{8}$ on its boundary with the vertex located in the origin $(0,0)$. We divide $\bar{\Omega}_i$ into a set of closed triangles $\{L_m\}$, where each L_m is a half of a closed square of the size h for corners $\omega_i, i = 1,2$; a half of a closed square of the size h in $\bar{\Omega}'_3 = [-1;1] \times [0;1]$ and a half of a closed rectangle with sizes h and $\frac{h}{2}$ in $\Omega_3 \setminus \bar{\Omega}'_3$ for a corner ω_3. Then, each L_m (macro-element) is refined (barycenter refinement) into three triangles K_{m_j}, their set $\{K_s\}$ (see Fig. 1).

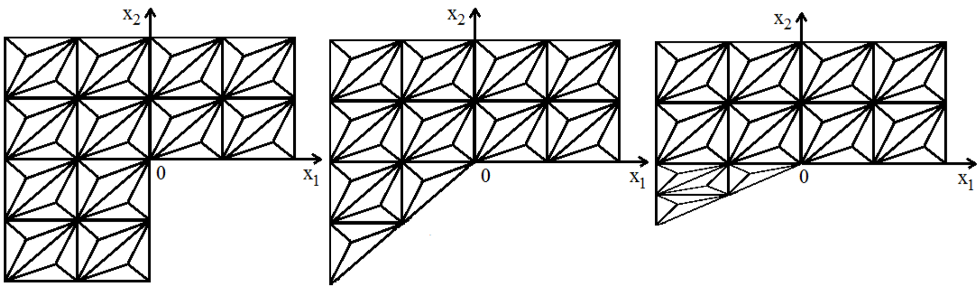

Fig. 1. The triangulation Υ_{ih} of the $\bar{\Omega}_i$: (left) $\omega_1 = \frac{3\pi}{2}$; (center) $\omega_2 = \frac{5\pi}{4}$; (right) $\omega_3 = \frac{9\pi}{8}$

We use the exact solution $(\mathbf{u}, p)$ of the Stokes problem (3), (4), which exhibits corner singularity phenomena at the reentrant corner ω_i on its boundary with the vertex located in the origin $(0,0)$. In polar coordinates (r, φ) at the origin the exact solution for each $\omega_i, i = 1,2,3$, is given by (see e.g. [25]):

$$u_1(r,\varphi) = r^{\lambda_i} \cdot ((1+\lambda_i)\,\Psi(\varphi)\cdot\sin(\varphi) + \Psi'(\varphi)\cdot\cos(\varphi)),$$

$$u_2(r,\varphi) = r^{\lambda_i} \cdot (\Psi'(\varphi)\cdot\sin(\varphi) - (1+\lambda_i)\,\Psi(\varphi)\cdot\cos(\varphi)),$$

$$p(r,\varphi) = -r^{\lambda_i - 1} \cdot \frac{(1+\lambda_i)^2\,\Psi'(\varphi) + \Psi'''(\varphi)}{1-\lambda_i},$$

$$\Psi(\varphi) = \frac{\sin((1+\lambda_i)\varphi)\cos(\lambda_i\omega_i)}{1+\lambda_i} - \cos((1+\lambda_i)\varphi) - \frac{\sin((1-\lambda_i)\varphi)\cos(\lambda_i\omega_i)}{1-\lambda_i} + \cos((1-\lambda_i)\varphi).$$

The exponent λ_i is the smallest positive solution of

$$\sin(\lambda\,\omega_i) + \lambda\sin(\omega_i) = 0,$$

which is $\lambda_1 \approx 0,544483$ for $\omega_1 = \frac{3\pi}{2}$, $\lambda_2 \approx 0,673583$ for $\omega_2 = \frac{5\pi}{4}$, $\lambda_3 \approx 0,800766$ for $\omega_3 = \frac{9\pi}{8}$. These solutions satisfy the Stokes problem (3), (4), where $\mathbf{f} \equiv 0$.

We emphasize that the pair $(\mathbf{u}, p)$ is analytical in $\bar{\Omega}_i \setminus (0,0)$, but $\nabla\mathbf{u}$ and p are singular at the origin. Especially, $\mathbf{u} \notin \mathbf{W}_2^2(\Omega)$ and $p \notin W_2^1(\Omega)$. This solution reflects perfectly the typical behaviour of the solution of the Stokes problem near a reentrant corner.

Table 1

The error norm $\|\mathbf{u}^h - \mathbf{u}\|_{\mathbf{W}_2^1(\Omega)}$ of the generalized solution, $\nu = 0$, $\delta = 1, \nu^* = \mu^* = 0$

ω_i	$N = 80$	$N = 160$	$N = 320$
$\frac{3\pi}{2}$	2,768e-1	1,898e-1	1,302e-1
$\frac{5\pi}{4}$	1,538e-1	9,649e-2	6,050e-2
$\frac{9\pi}{8}$	6,321e-2	3,627e-2	2,082e-2

Table 2

The influence of parameters δ and ν on the behaviour of the error $\|\mathbf{u}_\nu^h - \mathbf{u}\|_{\mathbf{W}_{2,\nu}^1(\Omega)}$ of the R_ν-generalized solution, $\nu^* = \mu^* = \lambda_i - 1$

ω_i	ν	δ	$N = 80$	$N = 160$	$N = 320$
$\frac{3\pi}{2}$	1,5	0,0125	2,619e-4	1,303e-4	6,475e-5
		0,015	3,946e-4	1,958e-4	9,744e-5
	1,8	0,0125	7,153e-5	3,551e-5	1,769e-5
		0,015	1,144e-4	5,709e-5	2,824e-5
$\frac{5\pi}{4}$	1,5	0,0125	1,362e-4	6,813e-5	3,404e-5
		0,015	2,140e-4	1,071e-4	5,332e-5
	1,8	0,0125	3,790e-5	1,886e-5	9,399e-6
		0,015	6,242e-5	3,107e-5	1,546e-5
$\frac{9\pi}{8}$	1,5	0,0125	7,581e-5	3,780e-5	1,880e-5
		0,015	9,826e-5	4,891e-5	2,428e-5
	1,8	0,0125	2,039e-5	1,017e-5	5,061e-6
		0,015	2,827e-5	1,409e-5	7,010e-6

Numerical experiments were carried out on meshes with different step sizes h (numbers $N, h = \frac{2}{N}$). The errors of the numerical approximations to the R_ν-generalized and generalized ($\nu = 0$, $\delta = 1, \nu^* = \mu^* = 0$) solutions were computed as the module between approximate and exact solutions in the points M_k and in the norm of spaces $W^1_{2,\nu}(\Omega)$ and $W^1_2(\Omega)$ respectively. The results of the numerical experiments are presented in Tables 1, 2. The optimal values of parameters ν and δ were derived numerically.

For the determined approximate R_ν-generalized and generalized solutions in Table 3 we present the numbers of points (in percentage of their total number), where the errors $\delta_{ji} = |u_j(M_i) - u^h_{\nu,j}(M_i)|, j = 1, 2, M_i \in R^{vel}_\Omega$ (for the R_ν-generalized solution) and $\delta'_{ji} = |u_j(M_i) - u^h_j(M_i)|, j = 1, 2, M_i \in R^{vel}_\Omega$ (for the generalized solution) are less than the given limit values $\bar{\triangle}_k$. In our experiments, the number of points (in percentage of their total number) for each component of the velocity field $\mathbf{u}$ is approximately the same.

Table 3

The number of points (in percentage of their total number), where the errors δ_{1i} and δ'_{1i} are less than the given limit values $\bar{\triangle}_k$

		R_ν-generalized, $\nu = 1,5$ $\delta = 0,0125, \nu^* = \lambda_i - 1$			generalized, $\nu = 0$ $\delta = 1, \nu^* = 0$		
ω_i	$\bar{\triangle}_k$	$N = 80$	$N = 160$	$N = 320$	$N = 80$	$N = 160$	$N = 320$
$\frac{3\pi}{2}$	10^{-4}	36,1%	46,5%	64,2%	31,4%	41,1%	52,3%
	10^{-5}	15,7%	18,9%	29,1%	14,9%	15,4%	23,1%
$\frac{5\pi}{4}$	10^{-4}	46,2%	59,2%	78,2%	41,3%	55,0%	70,9%
	10^{-5}	24,7%	29,8%	40,1%	20,8%	26,3%	35,2%
$\frac{9\pi}{8}$	10^{-4}	72,3%	78,5%	89,4%	68,4%	73,7%	84,7%
	10^{-5}	38,7%	49,0%	66,2%	33,8%	46,4%	63,2%

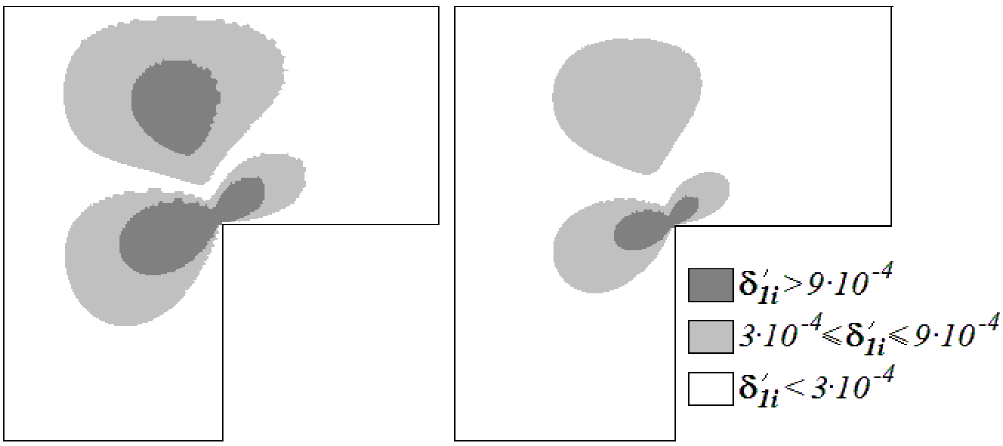

Fig. 2. Distribution of the points M_k with errors δ'_{1i} for the component u^h_1 of the approximate generalized solution $\omega_1 = \frac{3\pi}{2}, \nu = 0, \delta = 1, \nu^* = \mu^* = 0$, (left) with $N = 160$, (right) with $N = 320$

On Figs. 2, 3 we depict the distribution of the points M_i with errors δ_{1i} and δ'_{1i} for the components $u^h_{\nu,1}$ and u^h_1 of the approximate R_ν-generalized and generalized solutions,

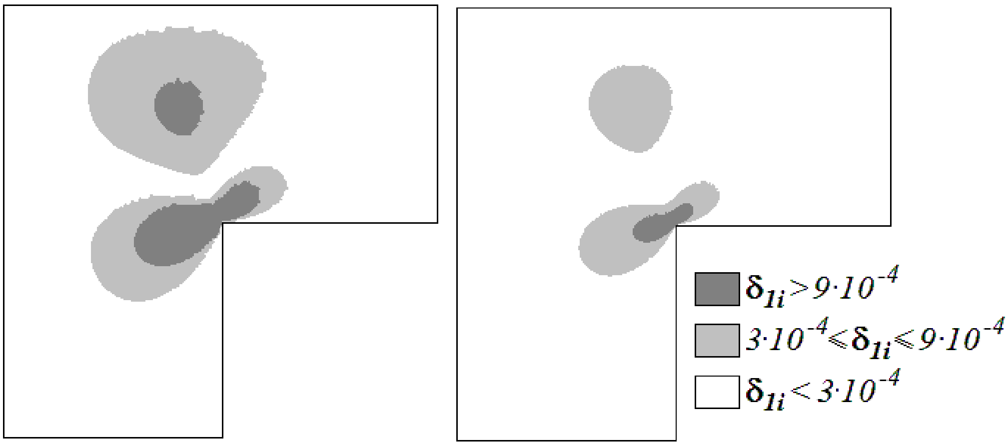

Fig. 3. Distribution of the points M_k with errors δ_{1i} for the component $u^h_{\nu,1}$ of the approximate R_ν-generalized solution $\omega_1 = \frac{3\pi}{2}, \nu = 1,5, \delta = 0,0125$, $\nu^* = \mu^* = \lambda_1 - 1$, (left) with $N = 160$, (right) with $N = 320$

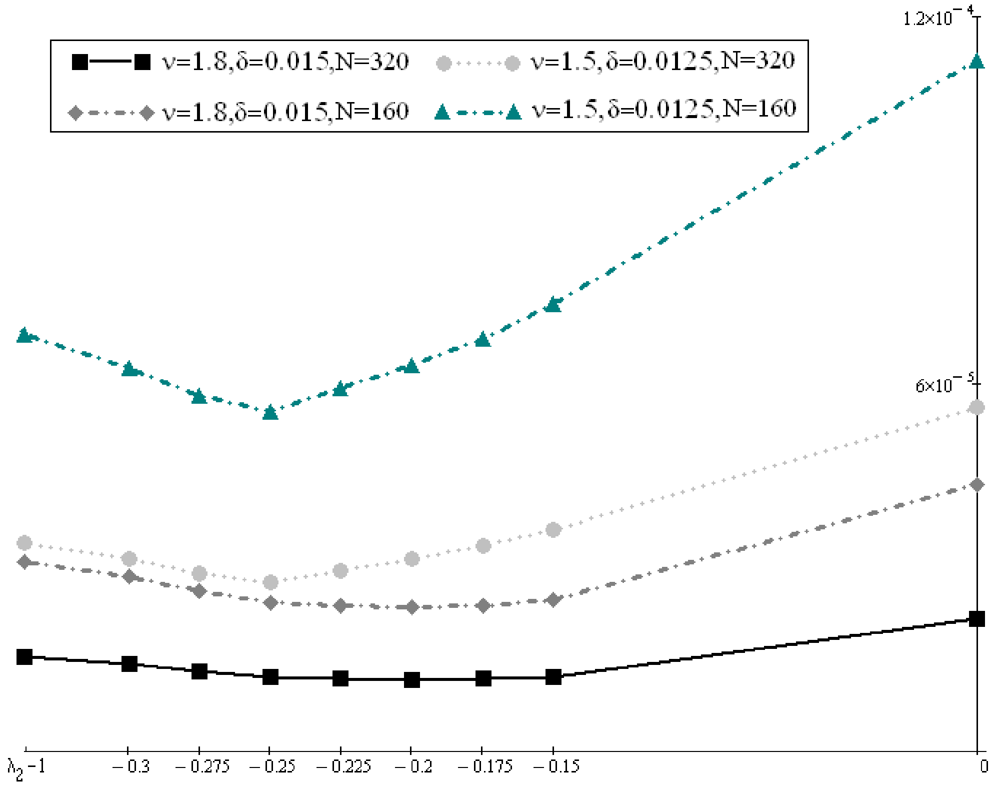

Fig. 4. The dependence of $\|\mathbf{u}^h_\nu - \mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}$ on the degree of ν^*, for $\omega_2 = \frac{5\pi}{4}$

respectively, for different mesh sizes h. On Figs. 4, 5 we introduce the dependence of errors in $\mathbf{W}^1_{2,\nu}(\Omega)$ norm on the degree $\nu^*(\mu^* = \nu^*)$ of the weight function $\rho(\mathbf{x})$. Each minimum on Figs. 4, 5 corresponds to the optimal value ν^* for the respective ν, δ and ω. This research was supported in through computational resources provided by the Shared Facility Center "Data Center of FEB RAS".

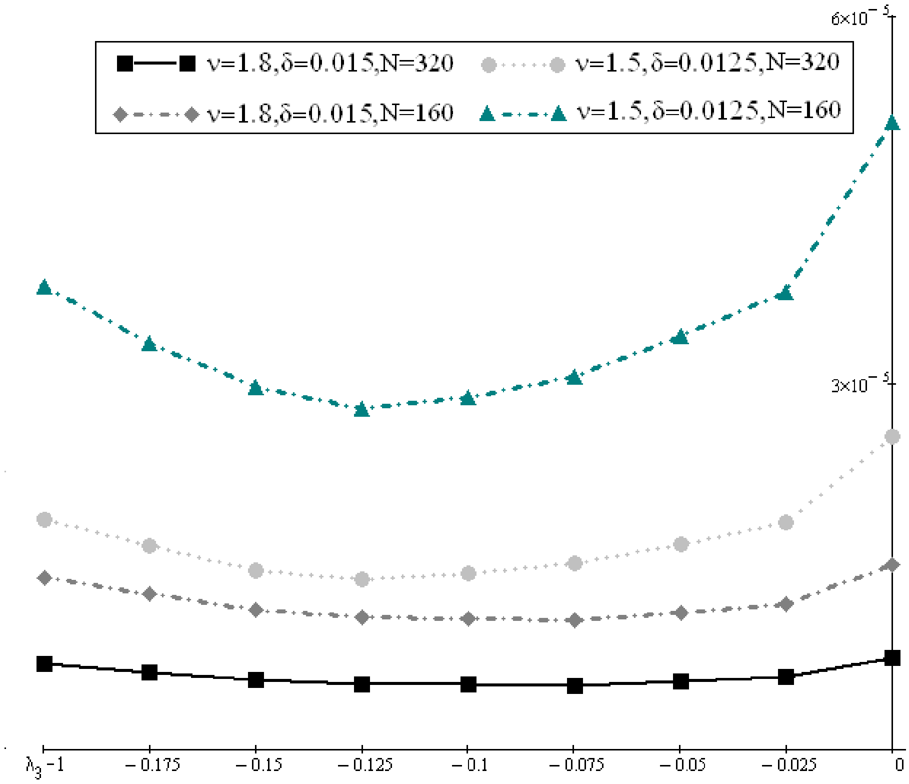

Fig. 5. The dependence of $\|\mathbf{u}_\nu^h - \mathbf{u}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}$ on the degree of ν^*, for $\omega_3 = \frac{9\pi}{8}$

Conclusions

The results of numerical experiments leads to the following conclusions:

- The approximate R_ν-generalized solution (velocity field) of the Stokes problem (3), (4) converges to the exact one with the rate $\mathcal{O}(h)$ in the $\mathbf{W}^1_{2,\nu}(\Omega)$ norm for all corners $\omega_i, i = 1, 2, 3$ (see Table 2), while the approximate generalized solution by classical FEM has an $\mathcal{O}(h^{0,55})$ rate of convergence for a corner $\omega_1 = \frac{3\pi}{2}$, $\mathcal{O}(h^{0,67})$ — for a corner $\omega_2 = \frac{5\pi}{4}$, $\mathcal{O}(h^{0,8})$ — for a corner $\omega_3 = \frac{9\pi}{8}$ in the $\mathbf{W}^1_2(\Omega)$ norm (see Table 1) (the so-called pollution effect [6]);
- The number of points in percentage of their total number, where the modulus of the difference between the approximate and exact solutions are less than the given limit values, more for the proposed weighted method in comparison with the classical FEM (see Table 3 and Figs. 2, 3);
- For all degrees $\nu^*(\mu^* = \nu^*)$ of the weight function $\rho(\mathbf{x})$, that lie between $\lambda_i - 1$ and 0, and parameters ν, δ close to optimal, the approximate R_ν-generalized solution converges to the exact one with the rate $\mathcal{O}(h)$ in the $\mathbf{W}^1_{2,\nu}(\Omega)$ norm (see Figs. 4, 5).

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НОВЫЙ ПРИБЛИЖЕННЫЙ МЕТОД РЕШЕНИЯ ЗАДАЧИ СТОКСА В ОБЛАСТИ С УГЛОВОЙ СИНГУЛЯРНОСТЬЮ

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В статье определено понятие R_ν-обобщенного решения задачи Стокса с сингулярностью в двумерной невыпуклой многоугольной области с одним входящим углом на границе области в специальных весовых множествах. Построено новое приближенное решение задачи с помощью весового метода конечных элементов. Предложен итерационный процесс решения полученной системы линейных алгебраических уравнений с блочным переобуславливанием ее матрицы на основе неполного алгоритма Удзавы и обобщенного метода минимальных невязок. Результаты численных экспериментов показали, что скорость сходимости приближенного R_ν-обобщенного решения к точному решению задачи не зависит от величины входящего угла на границе области и равна первой степени по шагу сетки h в норме весового пространства $W^1_{2,\nu}(\Omega)$ для компонент вектора скоростей, в отличие от стандартных конечно-элементных и конечно-разностных схем, приближенное решение которых сходится к точному решению задачи не быстрее чем со скоростью $\mathcal{O}(h^\alpha)$ в норме пространства $W^1_2(\Omega)$ для компонент вектора скоростей, где $\alpha < 1$ и степень α зависит от величины входящего угла.

Ключевые слова: угловая сингулярность; весовой метод конечных элементов; предобуславливатель.

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