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LORD KELVIN AND ANDREY ANDREYEVICH MARKOV IN A QUEUE WITH SINGLE SERVER

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We use Lord Kelvin's method of images to show that a certain infinite system of equations with interesting boundary conditions leads to a Markovian dynamics in an L^1-type space. This system originates from the queuing theory.

Keywords: queue; method of images; generation theorem; boundary conditions; Markovian dynamics.

Dedicated to Professor Jan Kisyński on the occasion of his 85th birthday.

Introduction

Background

As shown by an increasing body of examples, an intelligent enrichment of an underlying state space may change the nature of a problem under consideration and allow for its elegant solution. For instance, even the simplest scalar delay-differential equation

$$x'(t) = ax(t) + bx(t-\tau), \qquad t \geq 0, \tag{1}$$

where $a, b \in \mathbb{R}$ and $\tau > 0$ are given, does not lead to a well-posed problem if we think of $x(t) \in \mathbb{R}$ as a state of a process at time t and desire to describe dynamics in $\mathbb{R}$. One should rather think of the evolution of "history segments of x", i.e., say continuous, functions

$$u_t : [-\tau, 0] \to \mathbb{R}, \qquad t \geq 0,$$

defined by $u_t(s) = x(s+t), s \in [-\tau, 0]$ so that u_t contains the entire information on x in the time interval $[t-\tau, t]$. Then, (1) may be rewritten as a differential equation in the space $C[-\tau, 0]$ of continuous functions on $[-\tau, 0]$ and the related initial value (Cauchy) problem is well-posed. As it transpires, this rather simple idea leads to a quite satisfying theory and has far-reaching consequences (see [1, Section VI.6] or [2] and references given there).

For another instance, a time-inhomogeneous Markov process $(X(t))_{t\geq 0}$ in a state S becomes time-homogeneous if extended to the pair $(X(t), t)_{t\geq 0}$ and considered in the Cartesian product space $S \times \mathbb{R}^+$. Similarly, embedding the state-space of a non-Markovian process in a more suitable space may lead to Markovian dynamics (see e.g. [3]). Hidden Markov models, abounding in machine learning and biological sequence analysis [4, 5], are the other side of the same coin. Here, although the process is in fact Markovian from the very beginning, what we observe is only a part or shadow of its state-space.

A similar trick to that described for time-inhomogeneous Markov process allows reducing non-autonomous Cauchy problems to autonomous ones (see [1, Section VI.9, *Evolution Semigroups*]). However, interestingly, in contrast to the examples presented above, this transformation, presumably meant to simplify analysis, rarely helps in practice; in fact, I know of no particular instance were it is really useful.

The next example is at least intriguing. As developed by Samuel Goldstein [6] and Mark Kac [7], the stochastic process

$$\xi(t) = \int_0^t (-1)^{N_a(s)}\,\mathrm{d}s, \qquad t \ge 0,$$

where $N_a(t), t \ge 0$ is a Poisson process with expected value $\mathbb{E}\,N_a(t) = at$ (a is a positive constant), lies behind a probabilistic formula for telegraph equation. In [8] Jan Kisyński proves that the process

$$\left(\xi(t), (-1)^{N_a(t)}\right),$$

which is $\xi(t)$ "enriched" by the coordinate $(-1)^{N_a(t)}$, has independent increments in the non-commutative group $\mathbb{R} \times \{-1, 1\}$ with the following multiplication rule:

$$(\tau, k) \circ (\varsigma, l) = (\tau l + \varsigma, kl).$$

In other words, $\xi(t), t \ge 0$ is a Lévy process whereas $\left(\xi(t), (-1)^{N_a(t)}\right)$, $t \ge 0$ is a Markov process. While the former information suffices for a successful treatment of the telegraph equation (see e.g. [9, 10]), the latter gives an additional insight and allows for natural generalizations [8].

A Markovian Approach to a Single Server Queue

Our paper is devoted to an example of similar type originating from the queuing theory. It is well-known that unless quite restrictive conditions are imposed, a process of the form

$$N(t) = \#\text{ of customers in a queue at time } t$$

is non-Markovian [11]. However, as developed by e.g. D.R. Cox [12], in the case of M/G/1 queue (with Markov-type arrivals, general distribution of service time and one server), the two-dimensional process

$$(N(t), x(t)),\ t \ge 0, \tag{2}$$

where $x(t)$ is the time the customer being served has spent at a service point up to time t, is Markov. Recently, the latter idea has been reinvestigated by P. Gwiżdż in [13], where it was noted that the resulting process may be viewed as a piece-wise deterministic process of M.H.A. Davies [14–16] (see also [17]). Gwiżdż's paper contains (see Theorem 2.1 there) a proof of the fact that the system

$$\frac{\partial p_0(t)}{\partial t} = -\alpha p_0(t) + \int_0^\infty \mu(x) p_1(t,x)\,\mathrm{d}x \tag{3}$$

$$\frac{\partial p_1(t,x)}{\partial t} = -\frac{\partial p_1(t,x)}{\partial x} - (\alpha + \mu(x)) p_1(t,x), \tag{4}$$

$$\frac{\partial p_n(t,x)}{\partial t} = -\frac{\partial p_n(t,x)}{\partial x} - (\alpha + \mu(x)) p_n(t,x) + \alpha p_{n-1}(t,x),\ n \ge 2, x > 0, \tag{5}$$

(where $\alpha > 0$ is intensity of customers' arrivals, and $\mu(\cdot)$, a non-negative, bounded function is the hazard rate function for service time) supplemented with the following boundary conditions:

$$p_1(t,0) = \int_0^\infty p_2(t,x)\mu(x)\,\mathrm{d}x + \alpha p_0(t)$$
$$p_n(t,0) = \int_0^\infty p_{n+1}(t,x)\mu(x)\,\mathrm{d}x, \qquad n \geq 2, \tag{6}$$

has, for any initial data, a unique mild solution in a certain space of type L^1 (see further on for details). Here, $p_n(t,x)$ is the probability density that at time t there are $n \geq 1$ customers and the one being served has spent time x at the service point; $p_0(t)$ is the probability that at that time there are no customers in the queue. From the perspective of the theory of semigroups of operators, this theorem says that the related operator is the generator of a semigroup in this space. The latter semigroup governs the evolution of probability distributions of the process (2), and the fact that the semigroup is composed of Markov operators is expressed in the relation

$$p_0(t) + \sum_{n=1}^{\infty} \int_0^\infty p_n(t,x)\,\mathrm{d}x = 1,$$

which holds for all $t > 0$ provided it holds for $t = 0$.

Our Goal

Theorem 2.1 in [13] is obtained by a Greiner-like [18] domain-perturbation technique, extended to include unbounded domain-perturbations in L^1-type spaces, and designed in such a way that the perturbed semigroup remains positive if the original semigroup is positive (original Greiner's perturbation does not posses this feature). In this paper, we will show the same result more directly, using Lord Kelvin's method of images [19–24]. Here is the main idea of the proof.

If all the terms involving α or μ are removed from system (3)–(5), and boundary conditions (6) are disregarded, the resulting equations may be solved explicitly on the entire real line by the very simple formula:

$$p_0(t) = p_0(0), \quad p_n(t,x) = p_n(0, x-t), \qquad t \geq 0,\ x \in \mathbb{R},\ n = 1,2,\ldots, \tag{7}$$

provided $p_0(0)$ and $p_n(0,x)$ are known for all $x \in \mathbb{R}$. Since restoring all removed terms is a matter of a bounded perturbation, the question of solving (3)–(6) reduces to that of existence of a procedure which, given α and μ, assigns to $p_n(0,\cdot), n \geq 1$ defined on $\mathbb{R}^+$ their unique extensions to the entire $\mathbb{R}$ in such a way that functions (7), as restricted to $x \geq 0$, solve (3)–(5) (with appropriate terms removed) and, at the same time, boundary conditions (6) are satisfied. Note that although in the latter equations α and μ may be thought to be set to zero, they remain non-zero in the boundary conditions (6).

To prove that this idea works well is the aim of this paper.

1. The Main Theorem

Throughout the paper we assume that μ is a bounded, non-negative function on $\mathbb{R}^+$ and $\alpha > 0$ is a positive constant.

Let $L^1(\mathbb{R}^+)$ be the space of absolutely continuous functions on $\mathbb{R}^+$, and let

$$L := \mathbb{R} \times l^1(L^1(\mathbb{R}^+))$$

be the space of sequences $(f_n)_{n\geq 0}$ where f_0 is a real number and $f_n, n \geq 1$ are members of $L^1(\mathbb{R}^+)$, such that

$$\| (f_n)_{n\geq 0} \|_L := |f_0| + \sum_{n=1}^{\infty} \|f_n\|_{L^1(\mathbb{R}^+)} < \infty;$$

when equipped with this norm, L is a Banach space. In what follows, we will often use the following bounded linear functional on L:

$$\Sigma_I (f_n)_{n\geq 0} = f_0 + \sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\,\mathrm{d}x.$$

Let $\mathcal{D}$ be the set of $(f_n)_{n\geq 0} \in L$ such that

- each $f_n, n \geq 1$ is absolutely continuous with $f_n' \in L^1(\mathbb{R}^+)$,
- $\sum_{n=1}^{\infty} \|f_n'\|_{L^1(\mathbb{R}^+)} < \infty$.

Also, let $F_n : \mathcal{D} \to \mathbb{R}$ be the linear functionals given by

$$F_1 (f_m)_{m\geq 0} = f_1(0) - \int_0^{\infty} \mu(x) f_2(x)\,\mathrm{d}x - \alpha f_0, \tag{8}$$

$$F_n (f_m)_{m\geq 0} = f_n(0) - \int_0^{\infty} \mu(x) f_{n+1}(x)\,\mathrm{d}x, \qquad n \geq 2.$$

Here is the main theorem of the paper.

Theorem 1. *The operator $A_{\mu,\alpha}$ defined on the domain*

$$\mathcal{D}(A_{\mu,\alpha}) = \mathcal{D} \cap \bigcap_{n\geq 1} \ker F_n$$

by the formula

$$A_{\mu,\alpha} (f_n)_{n\geq 0} = (g_n)_{n\geq 0}\,,$$

where

$$g_0 = -\alpha f_0 + \int_0^{\infty} \mu(x) f_1(x)\,\mathrm{d}x,$$
$$g_1 = -f_1' - (\alpha + \mu) f_1,$$
$$g_n = -f_n' - (\alpha + \mu) f_n + \alpha f_{n-1}, \qquad n \geq 2,$$

generates a strongly continuous semigroup of Markov operators in L.

To recall, a bounded linear operator $P : L \to L$ is said to be a Markov operator, if $P (f_n)_{n\geq 0}$ is non-negative when $(f_n)_{n\geq 0}$ is, and for such $(f_n)_{n\geq 0}$ we have $\Sigma_I P (f_n)_{n\geq 0} = \Sigma_I (f_n)_{n\geq 0}$. If the latter equality is replaced by the inequality $\Sigma_I P (f_n)_{n\geq 0} \leq \Sigma_I (f_n)_{n\geq 0}$ the operator is said to be sub-Markov.

2. Reduction to a Simpler Case

We will show now that Theorem 1 may be deduced from the following result.

Theorem 2. *The operator $B_{\mu,\alpha}$ defined on the domain $\mathcal{D}(B_{\mu,\alpha}) = \mathcal{D}(A_{\mu,\alpha})$ by the formula*

$$B_{\mu,\alpha}\left(f_n\right)_{n\geq 0} = \left(g_n\right)_{n\geq 0},$$

where

$$g_0 = 0, \qquad g_n = -f_n', \qquad n \geq 1,$$

generates a strongly continuous semigroup of positive operators in L.

Remark 1. We stress again that, while α and μ are not featuring in the definition of the "action" of $B_{\mu,\alpha}$, both are involved in the definition of its domain. Hence, what we are facing here is a domain-changing perturbation; such perturbations were studied in detail by G. Greiner [18], and the approach presented in [13] follows Greiner's path. We are using a different method, i.e., Lord Kelvin's method of images that has been developed as a way to deal with boundary conditions in semigroup theory in [19–23] and [24]. For yet different ways of dealing with boundary conditions see [25] and [26].

Suppose thus that $B_{\mu,\alpha}$ generates a semigroup of positive operators and then choose a κ such that

$$\kappa \geq \sup_{x\geq 0} \mu(x).$$

Then, for any $\left(f_n\right)_{n\geq 0} \in \mathcal{D}(A_{\mu,\alpha}) = \mathcal{D}(B_{\mu,\alpha})$,

$$A_{\mu,\alpha}\left(f_n\right)_{n\geq 0} = B_{\mu,\alpha}\left(f_n\right)_{n\geq 0} + C_{\mu,\alpha,\kappa}\left(f_n\right)_{n\geq 0} - (\alpha+\kappa)\left(f_n\right)_{n\geq 0},$$

where $C_{\mu,\alpha,\kappa}$ is the bounded, non-negative, linear operator defined by

$$C_{\mu,\alpha,\kappa}\left(f_n\right)_{n\geq 0} = \left(g_n\right)_{n\geq 0}$$

with

$$g_0 = \int_0^\infty \mu(x) f_1(x)\,\mathrm{d}x + \kappa f_0, \qquad g_1 = (\kappa-\mu) f_1,$$
$$g_n = (\kappa-\mu) f_n + \alpha f_{n-1},\ n \geq 2.$$

Thus, by the Phillips perturbation theorem, $B_{\mu,\alpha}+C_{\mu,\alpha,\kappa}$ generates a semigroup of positive operators and this implies that so does $A_{\mu,\alpha}$, because the last two operators differ by a constant multiple of the identity operator.

Hence, we are left with showing that the semigroup generated by $A_{\mu,\alpha}$ is composed of Markov operators. A well-known necessary and sufficient condition for that is for $\lambda\left(\lambda - A_{\mu,\alpha}\right)^{-1}$ to be Markov operators for all sufficiently large $\lambda > 0$. Since $A_{\mu,\alpha}$ generates a semigroup, for sufficiently large λ the resolvent equation

$$\lambda\left(f_n\right)_{n\geq 0} - A_{\mu,\alpha}\left(f_n\right)_{n\geq 0} = \left(g_n\right)_{n\geq 0}$$

has a unique solution for all $(g_n)_{n\geq 0}$. Because the semigroup generated by $A_{\mu,\alpha}$ is composed of positive operators, the map assigning solution $(f_n)_{n\geq 0}$ to $(g_n)_{n\geq 0}$ is positive also (since the resolvent is the Laplace transform of the semigroup). Applying the functional Σ_I to both sides of the resolvent equation we see thus that all we need to prove is that

$$\Sigma_I A_{\mu,\alpha} (f_n)_{n\geq 0} = 0 \tag{9}$$

for all non-negative $(f_n)_{n\geq 0} \in \mathcal{D}(A_{\mu,\alpha})$.

To this end, we first use (8) to calculate (writing, for simplicity of notation, $\int\limits_{\mathbb{R}^+} \mu f_n$ instead of $\int\limits_0^\infty \mu(x) f_n(x)\,\mathrm{d}x$)

$$\sum_{n=1}^{\infty} f_n(0) = \alpha f_0 + \sum_{n=2}^{\infty} \int\limits_{\mathbb{R}^+} \mu_n f_n, \qquad (f_n)_{n\geq 0} \in \mathcal{D}(A_{\mu,\alpha}).$$

On the other hand, with similar notation,

$$\Sigma_I A_{\mu,\alpha} (f_n)_{n\geq 0} = -\alpha f_0 - \sum_{n=2}^{\infty} \int\limits_{\mathbb{R}^+} \mu_n f_n - \sum_{n=1}^{\infty} \int\limits_{\mathbb{R}^+} f_n'.$$

Since $\int\limits_{\mathbb{R}^+} f_n' = -f_n(0)$, these two relations combined imply (9), completing the proof.

Remark 2. The last calculation shows that (9) holds in fact for all vectors $(f_n)_{n\geq 0}$ in $\mathcal{D}(A_{\mu,\alpha})$ whether positive or not. Hence, $\lambda (\lambda - A_{\mu,\alpha})^{-1}$ preserves the functional Σ_I, i.e., $\Sigma_I \lambda (\lambda - A_{\mu,\alpha})^{-1} (f_n)_{n\geq 0} = \Sigma_I (f_n)_{n\geq 0}$ for all sufficiently large λ. Therefore, the semigroup generated by $A_{\mu,\alpha}$ also preserves this functional, i.e., the value of Σ_I does not change along this semigroup's trajectories (this is a stronger condition than that of being composed of Markov operators).

3. A Semigroup in a "Larger" Space

A first step in the procedure of the Lord Kelvin method of images is to construct a semigroup generated "by $B_{\mu,\alpha}$ without boundary conditions" in a "larger" space. This is our goal in this section.

Let $(f_n)_{n\geq 0} \in L$, and consider an $n \geq 1$. Any extension of f_n to a function $f_n^\diamondsuit$ on $\mathbb{R}$ is determined by $g_n(x) = f_n^\diamondsuit(-x)$, a function on $\mathbb{R}^+$. It would be nice to have $g_n \in L^1(\mathbb{R}^+)$ but this is rarely the case. Fortunately, for the particular extension we are looking for there is an $\omega \geq 0$ such that all functions $x \mapsto \mathrm{e}^{-\omega x} g_n(x)$ are in $L^1(\mathbb{R}^+)$. Hence, we introduce $L^1_\omega(\mathbb{R}^+)$ as the space of functions $g : \mathbb{R}^+ \to \mathbb{R}$ such that $x \mapsto \mathrm{e}^{-\omega x} g(x)$ is a member of $L^1(\mathbb{R}^+)$. When equipped with the norm

$$\|g\|_{L^1_\omega(\mathbb{R}^+)} = \int\limits_0^\infty |g(x)| \mathrm{e}^{-\omega x}\,\mathrm{d}x,$$

$L^1_\omega(\mathbb{R}^+)$ is a Banach space.

A pair $(f,g) \in L^1(\mathbb{R}^+) \times L^1_\omega(\mathbb{R}^+)$ may be identified with a function h on $\mathbb{R}$: it suffices to agree that $h(x) = f(x), x \geq 0$ and $h(x) = g(-x)$ for $x < 0$. The space $L^1_\omega(\mathbb{R})$ of functions $h : \mathbb{R} \to \mathbb{R}$ such that

$$\|h\|_{L^1_\omega(\mathbb{R})} := \int\limits_0^\infty |h(x)| \,\mathrm{d}x + \int\limits_0^\infty \mathrm{e}^{-\omega x} |h(-x)| \,\mathrm{d}x$$

is finite, is a Banach space with this norm, and the spaces $L^1(\mathbb{R}^+) \times L^1_\omega(\mathbb{R}^+)$ and $L^1_\omega(\mathbb{R})$ are isometrically isomorphic. All our extensions will be members of $L^1_\omega(\mathbb{R})$ and sequences of extensions will be members of

$$L_\omega := \mathbb{R} \times l^1(L^1_\omega(\mathbb{R})),$$

the space of sequences $(h_n)_{n\geq 0}$ where h_0 is a real number and $h_n, n \geq 1$, are members of $L^1_\omega(\mathbb{R})$, such that

$$\| \left(h_n\right)_{n\geq 0} \|_{L_\omega} := |h_0| + \sum_{n=1}^{\infty} \|h_n\|_{L^1_\omega(\mathbb{R})} < \infty;$$

when equipped with this norm, L_ω is a Banach space.

The formula

$$T(t)h(x) = h(x-t), \qquad t \geq 0,\, x \in \mathbb{R}, \tag{10}$$

defines a strongly continuous semigroup in $L^1_\omega(\mathbb{R})$, and since

$$\begin{aligned} \|T(t)h\|_{L^1_\omega(\mathbb{R})} &\leq \int\limits_0^t |h(x-t)| \,\mathrm{d}x + \int\limits_t^\infty |h(x-t)| \,\mathrm{d}x + \int\limits_0^\infty |h(-x-t)| \mathrm{e}^{-\omega x} \,\mathrm{d}x \\ &\leq \mathrm{e}^{\omega t} \int\limits_0^t \mathrm{e}^{-\omega x} |h(-x)| \,\mathrm{d}x + \int\limits_0^\infty |h(x)| \,\mathrm{d}x + \mathrm{e}^{\omega t} \int\limits_t^\infty \mathrm{e}^{-\omega x} |h(-x)| \,\mathrm{d}x \\ &\leq \mathrm{e}^{\omega t} \|h\|_{L^1_\omega(\mathbb{R})}, \end{aligned}$$

the operator norm of $T(t)$ does not exceed $\mathrm{e}^{\omega t}$. A standard argument shows that the domain $\mathcal{D}(G)$ of the generator G of $\{T(t), t \geq 0\}$ is composed of absolutely continuous members h of $L^1_\omega(\mathbb{R})$ such that $h' \in L^1_\omega(\mathbb{R})$, and $Gh = -h'$ on this domain.

It follows that

$$\mathcal{T}(t) \left(h_n\right)_{n\geq 0} = (h_0, T(t)h_1, T(t)h_2, \dots) \tag{11}$$

defines a strongly continuous semigroup of operators in L_ω. The operator norm of $\mathcal{T}(t)$ does not exceed $\mathrm{e}^{\omega t}$. The domain $\mathcal{D}(\mathcal{G})$ of the infinitesimal generator $\mathcal{G}$ of $\{\mathcal{T}(t), t \geq 0\}$ is composed of $(h_n)_{n\geq 0}$ such that $h_n, n \geq 1$ are absolutely continuous with $h_n' \in L^1_\omega(\mathbb{R})$ and $(0, h_1', h_2', \dots) \in L_\omega$. For such $(h_n)_{n\geq 0}$,

$$\mathcal{G} \left(h_n\right)_{n\geq 0} = -(0, h_1', h_2', \dots).$$

4. Constructing the Images

As explained in the introduction to [20], given an $(f_m)_{m\ge 0} \in \mathcal{D}(B_{\mu,\alpha})$ it is a good idea to look for the sequence $\left(f_m^{\diamondsuit}\right)_{m\ge 0} \in L_\omega$ of extensions of its terms satisfying

$$F_n \mathcal{T}(t)\left(f_m^{\diamondsuit}\right)_{m\ge 0} = 0, \qquad \text{for all } n \ge 1,\, t \ge 0, \tag{12}$$

where F_n given by (8) is now treated as a functional defined on $\mathcal{D}(\mathcal{G})$. This condition means that

$$f_1^{\diamondsuit}(-t) = \int_0^\infty \mu(x) f_2^{\diamondsuit}(x-t)\,\mathrm{d}x + \alpha f_0$$

and

$$f_n^{\diamondsuit}(-t) = \int_0^\infty \mu(x) f_{n+1}^{\diamondsuit}(x-t)\,\mathrm{d}x, \qquad \text{for all } n \ge 2, t \ge 0,$$

and thus may be rewritten as a requirement for

$$g_n(x) := f_n^{\diamondsuit}(-x), \qquad x \ge 0$$

as follows:

$$g_1(x) = \int_0^x g_2(x-y)\mu(y)\,\mathrm{d}y + \int_0^\infty \mu(x+y) f_2(y)\,\mathrm{d}y + \alpha f_0,$$
$$g_n(x) = \int_0^x g_{n+1}(x-y)\mu(y)\,\mathrm{d}y + \int_0^\infty \mu(x+y) f_{n+1}(y)\,\mathrm{d}y, \quad n \ge 2, x \ge 0.$$

This in turn simply means that we need to have

$$g_1 = \mu * g_2 + Uf_2 + \alpha f_0, \qquad g_n = \mu * g_{n+1} + Uf_{n+1}, \qquad n \ge 2, \tag{13}$$

where $Uf(x) = \int_0^\infty \mu(x+y) f(y)\,\mathrm{d}y, f \in L^1(\mathbb{R}^+)$.

We note that while condition (12) may be formulated only for $(f_n)_{n\ge 0}$ in $\mathcal{D}(B_{\mu,\alpha})$, (13) makes sense for all $(f_n)_{n\ge 0}$ in L.

Lemma 1. *Fix $\omega > \|\mu\|_\infty := \sup_{x\ge 0} \mu(x)$. For each $(f_n)_{n\ge 0} \in L$ there is precisely one $(g_n)_{n\ge 1}$ such that (13) holds and the sequence $\left(f_n^{\diamondsuit}\right)_{n\ge 0}$ defined by*

$$f_0^{\diamondsuit} = f_0,$$
$$f_n^{\diamondsuit}(x) = f_n(x), \qquad x \ge 0,$$
$$f_n^{\diamondsuit}(x) = g_n(-x), \qquad x < 0,$$

is a member of L_ω.

Proof.

1. Let $K_\omega := l^1(L^1_\omega(\mathbb{R}^+))$ be the space of sequences $(k_n)_{n\ge 1}$ such that $k_n \in L^1_\omega(\mathbb{R}^+)$ and

$$\|\,(k_n)_{n\ge 1}\,\|_{K_\omega} := \sum_{n=1}^\infty \|k_n\|_{L^1_\omega(\mathbb{R}^+)} < \infty.$$

Our task is thus to show that for any $(f_n)_{n\geq 0} \in L$, there is precisely one $(g_n)_{n\geq 1} \in K_\omega$ satisfying (13).

2. We start the proof by noting that Uf defined below (13) belongs to $L^1_\omega(\mathbb{R}^+)$ and

$$\|Uf\|_{L^1_\omega(\mathbb{R}^+)} \leq \frac{\|\mu\|_\infty}{\omega}\|f\|_{L^1(\mathbb{R}^+)}. \tag{14}$$

This is showed by the following calculation:

$$\|Uf\|_{L^1_\omega(\mathbb{R}^+)} \leq \int\limits_0^\infty \mathrm{e}^{-\omega x} \int\limits_0^\infty \mu(x+y)|f(y)|\,\mathrm{d}y\,\mathrm{d}x \leq$$
$$\leq \int\limits_0^\infty \mathrm{e}^{-\omega x}\|\mu\|_\infty \int\limits_0^\infty |f(y)|\,\mathrm{d}y\,\mathrm{d}x = \frac{\|\mu\|_\infty}{\omega}\|f\|_{L^1(\mathbb{R}^+)}.$$

Inequality (14) implies that

$$\mathcal{U}\,(f_n)_{n\geq 0} := (Uf_2 + \alpha f_0, Uf_3, Uf_4, ...)$$

is a member of K_ω with

$$\|\mathcal{U}\,(f_n)_{n\geq 0}\|_{K_\omega} \leq \sum_{n=2}^{\infty} \|Uf_n\|_{L^1_\omega(\mathbb{R}^+)} + \frac{\alpha|f_0|}{\omega} \leq \frac{\max(\|\mu\|_\infty, \alpha)}{\omega}\|\,(f_n)_{n\geq 0}\,\|_L. \tag{15}$$

3. Finally, for a fixed $(f_n)_{n\geq 0} \in L$, let $\mathcal{M} : K_\omega \to K_\omega$ be the map given by

$$\mathcal{M}\,(k_n)_{n\geq 1} = (\mu * k_{n+1})_{n\geq 1} + \mathcal{U}\,(f_n)_{n\geq 0}\,. \tag{16}$$

For any $k \in L^1_\omega(\mathbb{R}^+)$,

$$\|\mu * k\|_{L^1_\omega(\mathbb{R}^+)} \leq \|\mu\|_{L^1_\omega(\mathbb{R}^+)}\,\|k\|_{L^1_\omega(\mathbb{R}^+)} \leq \frac{\|\mu\|_\infty}{\omega}\,\|k\|_{L^1_\omega(\mathbb{R}^+)}. \tag{17}$$

It follows that

$$\|\,(\mu * k_{n+1})_{n\geq 1}\,\|_{K_\omega} \leq \frac{\|\mu\|_\infty}{\omega}\,\|\,(k_n)_{n\geq 1}\,\|_{K_\omega}. \tag{18}$$

Since $\|\mu\|_\infty < \omega$, $\mathcal{M}$ is a contraction mapping and Banach's fixed point theorem implies that there is precisely one $(g_n)_{n\geq 1} \in K_\omega$ such that $\mathcal{M}\,(g_n)_{n\geq 1} = (g_n)_{n\geq 1}$, i.e. precisely one $(g_n)_{n\geq 1} \in K_\omega$ satisfying (13). □

The sequence $(g_n)_{n\geq 1}$, whose existence has just been established, will be called the (μ, α)-image of $(f_n)_{n\geq 0}$, and $\left(f_n^\diamondsuit\right)_{n\geq 0}$ will be called the (μ, α)-extension of $(f_n)_{n\geq 0}$. We note that for non-negative $(f_n)_{n\geq 0}$, the vector $\mathcal{U}\,(f_n)_{n\geq 0}$ is also non-negative, and thus the map $\mathcal{M}$ defined in (16) leaves the non-negative cone in K_ω invariant. Since the fix-point of $\mathcal{M}$ may be obtained as $\lim\limits_{l\to\infty} \mathcal{M}^l\,(k_n)_{n\geq 1}$ for any $(k_n)_{n\geq 1} \in K_\omega$ and in particular we may take $(k_n)_{n\geq 1} = \mathcal{U}\,(f_n)_{n\geq 0}$ it follows that $(g_n)_{n\geq 1}$ is non-negative provided $(f_n)_{n\geq 0}$ is.

It is worth noting that (18) forces

$$\|\mathcal{M}\,(k_n)_{n\geq 1}\,\|_{K_\omega} \leq \frac{\|\mu\|_\infty}{\omega}\|\,(k_n)_{n\geq 1}\,\|_{K_\omega} + \|\mathcal{U}\,(f_n)_{n\geq 0}\,\|_{K_\omega}.$$

Thus, by induction,

$$\|\mathcal{M}^\ell \mathcal{U}\,(f_n)_{n\ge 0}\,\|_{K_\omega} \le \sum_{i=0}^{\ell}\left(\frac{\|\mu\|_\infty}{\omega}\right)^i \|\mathcal{U}\,(f_n)_{n\ge 0}\,\|_{K_\omega}, \qquad \ell \ge 1,$$

and so, using (15) and $(g_n)_{n\ge 1} = \lim_{\ell\to\infty} \mathcal{M}^\ell \mathcal{U}\,(f_n)_{n\ge 0}$,

$$\|\,(g_n)_{n\ge 1}\,\|_{K_\omega} \le \frac{\omega}{\omega - \|\mu\|_\infty}\|\mathcal{U}\,(f_n)_{n\ge 0}\,\|_{K_\omega} \le \frac{\max(\|\mu\|_\infty, \alpha)}{\omega - \|\mu\|_\infty}\|\,(f_n)_{n\ge 0}\,\|_{L}. \tag{19}$$

We also need information on regularity of (μ, α)-images, contained in Lemma 3 (see later on) which in turn is based on the following Sobolev type inequality.

Lemma 2. *Let $\omega \ge 0$. Suppose $f \in L^1_\omega(\mathbb{R}^+)$ is absolutely continuous with $f' \in L^1_\omega(\mathbb{R}^+)$. Then*

$$|f(0)| \le \omega\|f\|_{L^1_\omega(\mathbb{R}^+)} + \|f'\|_{L^1_\omega(\mathbb{R}^+)}.$$

Proof. The function $x \mapsto \mathrm{e}^{-\omega x} f(x)$ is also absolutely continuous with derivative equal to $\mathrm{e}^{-\omega x} f'(x) - \omega \mathrm{e}^{-\omega x} f(x)$. For $x \ge 0$ we have thus

$$\mathrm{e}^{-\omega x} f(x) - f(0) = \int_0^x [\mathrm{e}^{-\omega y} f(y)]'\,\mathrm{d}y.$$

Since the limit as $x \to \infty$ of the right-hand side exists, so must the limit $\lim_{x\to\infty} \mathrm{e}^{-\omega x} f(x)$. But the latter must be zero, $x \mapsto \mathrm{e}^{-\omega x} f(x)$ being integrable. Therefore,

$$|f(0)| = \left|\int_0^\infty [\omega \mathrm{e}^{-\omega y} f(y) - \mathrm{e}^{-\omega y} f'(y)]\,\mathrm{d}y\right| \le \omega\|f\|_{L^1_\omega(\mathbb{R}^+)} + \|f'\|_{L^1_\omega(\mathbb{R}^+)},$$

as desired. □

Lemma 3. *Fix $(f_n)_{n\ge 1} \in \mathcal{D}$ (see Section 1). Then $g_n, n \ge 1$ are absolutely continuous with $g_n' \in L^1_\omega(\mathbb{R}^+)$ and*

$$\|\,(g_n')_{n\ge 1}\,\|_{K_\omega} = \sum_{n=1}^\infty \|g_n'\|_{L^1_\omega(\mathbb{R}^+)} < \infty.$$

Proof.

1. Suppose $f \in L^1(\mathbb{R}^+)$ is absolutely continuous with $f' \in L^1(\mathbb{R}^+)$. Then Uf (defined right after (13)) is absolutely continuous also, and

$$-(Uf)' = Uf' + f(0)\mu.$$

2. Let $(k_n)_{n\ge 1} := \mathcal{U}\,(f_n)_{n\ge 0} \in K_\omega$. By point 1, each k_n is absolutely continuous with $-k_n' = Uf_{n+1}' + f_{n+1}(0)\mu$. Therefore, by Lemma 2 with $\omega = 0$ and (14),

$$\sum_{n=1}^\infty \|k_n'\|_{L^1_\omega(\mathbb{R}^+)} \le \sum_{n=2}^\infty \|Uf_n'\|_{L^1_\omega(\mathbb{R}^+)} + \|\mu\|_{L^1_\omega(\mathbb{R}^+)} \sum_{n=2}^\infty |f_n(0)| \le$$

$$\le \frac{\|\mu\|_\infty}{\omega}\sum_{n=2}^\infty \|f_n'\|_{L^1(\mathbb{R}^+)} + \frac{\|\mu\|_\infty}{\omega}\sum_{n=2}^\infty \|f_n'\|_{L^1(\mathbb{R}^+)} < \infty.$$

Combining this with (15) we see that

$$\| (k_n)_{n\geq 1} \|_{K_\omega} + \| (k'_n)_{n\geq 1} \|_{K_\omega} < \infty. \tag{20}$$

3. An induction argument shows that the n-th coordinate of $\mathcal{M}^\ell (k_n)_{n\geq 1}$ is $\sum\limits_{i=0}^{\ell} \mu^{i*} * k_{i+n}, \ell \geq 1$. Since $(g_n)_{n\geq 1} = \lim\limits_{\ell\to\infty} \mathcal{M}^\ell (k_n)_{n\geq 1}$ we have

$$g_n = \sum_{i=0}^{\infty} \mu^{i*} * k_{i+n}, \qquad n \geq 1. \tag{21}$$

4. If a $k \in L^1_\omega(\mathbb{R}^+)$ is absolutely continuous with $k' \in L^1_\omega(\mathbb{R}^+)$, then so is $\mu * k$ and $(\mu * k)' = \mu * k' + k(0)\mu$. Since $(\mu^{i*} * k)(0) = 0$ for $i \geq 1$, an induction argument shows that

$$(\mu^{i*} * k)' = \mu^{i*} * k' + k(0)\mu^{i*}, \qquad i \geq 1.$$

This in turn yields

$$\left(\sum_{i=0}^{\ell} \mu^{i*} * k_{i+n} \right)' = \sum_{i=0}^{\ell} \mu^{i*} * k'_{i+n} + \sum_{i=1}^{\ell} k_{i+n}(0)\mu^{i*}.$$

5. We claim that

$$\sum_{n=1}^{\infty} \left(\sum_{i=0}^{\infty} \|\mu^{i*} * k'_{i+n}\|_{L^1_\omega(\mathbb{R}^+)} + \sum_{i=1}^{\infty} \|k_{i+n}(0)\mu^{i*}\|_{L^1_\omega(\mathbb{R}^+)} \right) < \infty. \tag{22}$$

Since (use (17))

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=0}^{\infty} \|\mu^{i*} * k'_{i+n}\|_{L^1_\omega(\mathbb{R}^+)} &\leq \sum_{n=1}^{\infty}\sum_{i=0}^{\infty} \left(\frac{\|\mu\|_\infty}{\omega}\right)^i \|k'_{i+n}\|_{L^1_\omega(\mathbb{R}^+)} \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\|\mu\|_\infty}{\omega}\right)^i \sum_{n=1}^{\infty} \|k'_n\|_{L^1_\omega(\mathbb{R}^+)} \\ &= \frac{\omega}{\omega - \|\mu\|_\infty} \| (k'_n)_{n\geq 1} \|_{K_\omega}\end{aligned}$$

is finite by (20), to prove (22) we need to estimate

$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \|k_{i+n}(0)\mu^{i*}\|_{L^1_\omega(\mathbb{R}^+)}.$$

By Lemma 2, however, this quantity does not exceed

$$\begin{aligned}&\sum_{i=1}^{\infty} \left(\frac{\|\mu\|_\infty}{\omega}\right)^i \sum_{n=1}^{\infty} \left[\omega \|k_n\|_{L^1_\omega(\mathbb{R}^+)} + \|k'_n\|_{L^1_\omega(\mathbb{R}^+)}\right] = \\ &= \frac{\|\mu\|_\infty}{\omega - \|\mu\|_\infty} \left(\omega \| (k_n)_{n\geq 1} \|_{K_\omega} + \| (k'_n)_{n\geq 1} \|_{K_\omega}\right)\end{aligned}$$

and this is finite by (20). This completes the proof of (22).

6. Let $\mathcal{D}(D) \subset L^1_\omega(\mathbb{R}^+)$ be the set of absolutely continuous $f \in L^1_\omega(\mathbb{R}^+)$ such that $f' \in L^1_\omega(\mathbb{R}^+)$, and let $Df = f'$. A straightforward argument shows that D is closed. Inequality (22) implies that for $n \geq 1$, the series

$$k'_n + \sum_{i=1}^{\infty} \left[\mu^{i*} * k'_{i+n} + k_{i+n}(0)\mu^{i*}\right] \tag{23}$$

converges absolutely in $L^1_\omega(\mathbb{R}^+)$. It follows that $D(\sum_{i=0}^{\ell} \mu^{i*} * k_{i+n})$ (calculated in point 4.) converges to the sum of this series. Since D is closed, g_n (given by (21)) belongs to $\mathcal{D}(D)$, i.e., it is absolutely continuous with g'_n equal to the sum of the series (23). Combining

$$\|g'_n\|_{L^1_\omega(\mathbb{R}^+)} \leq \sum_{i=0}^{\infty} \|\mu^{i*} * k'_{i+n}\|_{L^1_\omega(\mathbb{R}^+)} + \sum_{i=1}^{\infty} \|k_{i+n}(0)\mu^{i*}\|_{L^1_\omega(\mathbb{R}^+)}$$

and (22), we complete the proof. □

5. Abstract Kelvin Formula for $\{e^{tB_{\mu,\alpha}}, t \geq 0\}$

The stage is now ready for the proof of Theorem 2. In fact, we will show the following result, giving a somewhat deeper insight into the nature of the semigroup generated by $B_{\mu,\alpha}$.

Theorem 3. *Let $R : L^1_\omega(\mathbb{R}) \to L^1(\mathbb{R}^+)$ map an $f \in L^1_\omega(\mathbb{R})$ to its restriction $f_{|[0,\infty)}$, and let $\mathcal{R} : L_\omega \to L$ be defined by*

$$\mathcal{R}\,(f_n)_{n\geq 0} = (f_0, Rf_1, Rf_2, Rf_3, \dots).$$

Also, let $\mathcal{E} : L \to L_\omega$ map elements of L to their (μ, α)-extensions. The **abstract Kelvin formula**

$$\mathcal{S}(t) = \mathcal{R}\mathcal{T}(t)\mathcal{E}, \qquad t \geq 0, \tag{24}$$

where $\{\mathcal{T}(t), t \geq 0\}$ is the translation semigroup of (11), defines a strongly continuous semigroup of operators in L, and the infinitesimal generator of $\{\mathcal{S}(t), t \geq 0\}$ is $B_{\mu,\alpha}$.

To see that Theorem 2 is a direct consequence of Theorem 3 it suffices to recall from the previous section that (μ, α)-extensions of non-negative $(f_n)_{n\geq 0}$ are non-negative, and that $\{\mathcal{T}(t), t \geq 0\}$ is a semigroup of non-negative operators.

The proof of Theorem 3 will become more clear if we extract from it the following lemma.

Lemma 4. *Let $(f_n)_{n\geq 0} \in \mathcal{D}(B_{\mu,\alpha})$ be fixed.*

(a) $\left(f_n^\diamondsuit\right)_{n\geq 0}$, the (μ, α)-extension of $(f_n)_{n\geq 0}$, belongs to the domain $\mathcal{D}(\mathcal{G})$ of the infinitesimal generator of the translation semigroup (11),

(b) For all $t \geq 0, \mathcal{T}(t)(f_n)_{n\geq 0}$ is the (μ, α)-extension of its own restriction $\mathcal{R}\mathcal{T}(t)(f_n)_{n\geq 0}$.

Proof. (a) By Lemma 3, our assumption implies that each $g_n, n \geq 1$ is absolutely continuous with $g_n' \in L^1_\omega(\mathbb{R}^+)$. Moreover, by definition of g_n,

$$g_1(0) = Uf_2(0) + \alpha f_0 = \int_0^\infty \mu(y) f_2(y)\,\mathrm{d}y + \alpha f_0$$

$$g_n(0) = Uf_{n+1}(0) = \int_0^\infty \mu(y) f_{n+1}(y)\,\mathrm{d}y, \qquad n \geq 2. \tag{25}$$

Since $(f_m)_{m\geq 0} \in \bigcap_{n\geq 1} \ker F_n,\ f_n(0) = g_n(0)$ and so $f_n^\diamondsuit$ is absolutely continuous for all $n \geq 1$. Moreover, by

$$\|(f_n^\diamondsuit)'\|_{L^1_\omega(\mathbb{R})} = \|f_n'\|_{L^1(\mathbb{R}^+)} + \|g_n'\|_{L^1_\omega(\mathbb{R}^+)},$$

Lemma 3 implies

$$\sum_{n=1}^\infty \|(f_n^\diamondsuit)'\|_{L^1_\omega(\mathbb{R})} < \infty,$$

completing the proof of (a).

(b) Fix $t \geq 0$. By (a), $\left(f_n^\diamondsuit\right)_{n\geq 0}$ is a member of $\mathcal{D}(\mathcal{G})$. It follows that so is $\left(f_n^\clubsuit\right)_{n\geq 0} := \mathcal{T}(t)\left(f_n^\diamondsuit\right)_{n\geq 0}$ and that $F_n\mathcal{T}(s)\left(f_m^\diamondsuit\right)_{m\geq 0} = 0$ for all $s \geq 0$ and $n \geq 1$ (on $\mathcal{D}(\mathcal{G})$, (12) and (13) are equivalent). Therefore, for all s and $n \geq 1$, $F_n\mathcal{T}(s)\left(f_m^\clubsuit\right)_{m\geq 0} = F_n\mathcal{T}(s+t)\left(f_m^\diamondsuit\right)_{m\geq 0} = 0$. This means, by definition, that $\left(f_m^\clubsuit\right)_{m\geq 0}$ is the (μ,α)-extension (of its own restriction). □

Proof of Theorem 3

1. Fix $\omega > \|\mu\|_\infty$, and let $E_\omega \subset L_\omega$ be the space of (μ,α)-extensions of members of L. Inequality (19) shows that $\mathcal{E}$ mapping L onto E_ω is bounded. Since $\mathcal{E}$ has a bounded inverse $\mathcal{R}$, E_ω is closed in L_ω, and hence is a Banach space (with norm inherited from L_ω). The spaces L and E_ω are isomorphic with the isomorphism $\mathcal{E} : L \to E_\omega$ and its inverse $\mathcal{R} : E_\omega \to L$.

2. Since $\mathcal{D}(B_{\mu,\alpha})$ is dense in L (as a straightforward argument shows), so is its image $\mathcal{E}\mathcal{D}(B_{\mu,\alpha})$ in E_ω. Lemma 4 now says that $\mathcal{E}\mathcal{D}(B_{\mu,\alpha})$ is invariant for the translation semigroup $\{\mathcal{T}(t), t \geq 0\}$. It follows that so is E_ω. Hence, $\{\mathcal{T}(t), t \geq 0\}$ restricted to E_ω is a strongly continuous semigroup. The semigroup defined by the abstract Kelvin formula (24) is thus the isomorphic image of $\{\mathcal{T}(t), t \geq 0\}$ restricted to E_ω, and it is obviously strongly continuous.

3. We are left with showing that the generator of $\{\mathcal{S}(t), t \geq 0\}$ is $B_{\mu,\alpha}$. To this end, we recall that the generator of $\{\mathcal{T}(t), t \geq 0\}$ restricted to E_ω is the part $\mathcal{G}_\mathrm{p}$ of $\mathcal{G}$ in E_ω ($\mathcal{G}$ was defined in Section 3). Thus $\left(f_n^\diamondsuit\right)_{n\geq 0} \in E_\omega$ is a member of $\mathcal{D}(\mathcal{G}_\mathrm{p})$ $(= \mathcal{D}(\mathcal{G}) \cap E_\omega)$ iff $f_n^\diamondsuit, n \geq 1$ are absolutely continuous and $(0, (f_1^\diamondsuit)', (f_2^\diamondsuit)', \dots) \in L_\omega$; then

$$\mathcal{G}_\mathrm{p}\left(f_n^\diamondsuit\right)_{n\geq 0} = -(0, (f_1^\diamondsuit)', (f_2^\diamondsuit)', \dots);$$

the vector on the right-hand side here automatically belongs to E_ω since E_ω is invariant for the translation semigroup.

On the other hand, $(f_n)_{n\geq 0}$ belongs to the domain of the generator, say $\mathcal{G}_1$, of $\{\mathcal{S}(t), t \geq 0\}$ iff $\left(f_n^\diamondsuit\right)_{n\geq 0} = \mathcal{E}\,(f_n)_{n\geq 0}$ belongs to $\mathcal{D}(\mathcal{G}_\mathrm{p})$. Lemma 4 tells us that for $(f_n)_{n\geq 0} \in \mathcal{D}(B_{\mu,\alpha})$

the latter condition holds. Conversely, if $\left(f_n^{\diamondsuit}\right)_{n\geq 0}$ is a member of $\mathcal{D}(\mathcal{G}) \cap E_\omega$, then each f_n (being the restriction of $f_n^{\diamondsuit}$) must be absolutely continuous with $f_n' \in L^1(\mathbb{R}^+)$, and we must have

$$\sum_{n=1}^{\infty} \|f_n'\|_{L^1(\mathbb{R}^+)} \leq \sum_{n=1}^{\infty} \|(f_n^{\diamondsuit})'\|_{L^1_\omega(\mathbb{R})} < \infty.$$

Also, absolute continuity of $f_n^{\diamondsuit}$ implies $f_n(0) = g_n(0)$ for all n, and then a look at (25) reveals that $(f_n)_{n\geq 0} \in \bigcap_{n\geq 1} \ker F_n$, thus showing that $\mathcal{D}(\mathcal{G}_1) = \mathcal{D}(B_{\mu,\alpha})$.

For such $(f_n)_{n\geq 0}$,

$$\mathcal{G}_1 \left(f_n\right)_{n\geq 0} = \mathcal{R}\mathcal{G}\mathcal{E} \left(f_n\right)_{n\geq 0} = \mathcal{R}\mathcal{G} \left(f_n^{\diamondsuit}\right)_{n\geq 0} =$$
$$= -\mathcal{R}(0, (f_1^{\diamondsuit})', (f_2^{\diamondsuit})', \dots) = -(0, f_1', f_2', \dots) = B_{\mu,\alpha} \left(f_n\right)_{n\geq 0}.$$

This completes the proof. □

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ЛОРД КЕЛЬВИН И АНДРЕЙ АНДРЕЕВИЧ МАРКОВ К ОЧЕРЕДИ С ОДИНОЧНОГО СЕРВЕРА

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Мы используем метод изображений лорда Кельвина, чтобы показать, что некоторая бесконечная система уравнений с интересными граничными условиями приводит к марковской динамике в пространстве L^1-типа. Эта система берет свое начало в теории массового обслуживания.

Ключевые слова: очередь; метод изображений; теорема генерации; граничные условия; Марковская динамика.

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