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ABOUT NONUNIQUENESS OF SOLUTIONS OF THE SHOWALTER–SIDOROV PROBLEM FOR ONE MATHEMATICAL MODEL OF NERVE IMPULSE SPREAD IN MEMBRANE

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The article is devoted to the study of the morphology of the phase space of a mathematical model of the nerve impulse spread in a membrane, based on a degenerate Fitz Hugh–Nagumo system, defined on a bounded domain with a smooth boundary. In this mathematical model, the rate of change of one of the components of the system can significantly exceed the other, which leads to a degenerate Fitz Hugh–Nagumo system. The model under inquiry belongs to a wide class of semilinear Sobolev type models. To research the problem of nonuniqueness of solutions of the Showalter–Sidorov problem, the phase space method will be used, which was developed by G.A. Sviridyuk to scrutinize the solvability of Sobolev type equations. We have shown that the phase space of the studied model contains singularity such as the Whitney fold. The conditions of existence, uniqueness or multiplicity of solutions of the Showalter–Sidorov problem depending on the parameters of the system are found.

Keywords: Sobolev type equations; Showalter–Sidorov problem; Fitz Hugh–Nagumo system; nonuniqueness of the solution.

Dedicated to Professor Viktor Chistyakov on the occasion of his 70th birthday.

Introduction

An important part of the development of modern biophysics is the study of mathematical models of processes in living nature. Processes such as blood clotting, nerve impulse spreading, cardiac muscle contraction can be modelled using the Fitz Hugh–Nagumo system of equations [1, 2]:

$$\begin{cases} \epsilon_1 v_t = \alpha_1 v_{ss} + \beta_1 w - \kappa_1 v, \\ \epsilon_2 w_t = \alpha_2 w_{ss} + \beta_2 w - \kappa_2 v - w^3, \end{cases} \tag{1}$$

where parameters α_1, α_2, β_1, $\kappa_1 \in \mathbb{R}_+$, β_2, $\kappa_2 \in \mathbb{R}$, $\epsilon_1, \epsilon_2 \geq 0$. System (1), on the one hand, is the development of the classical Kolmogorov–Petrovsky–Piskunov model, and on the other hand, some simplified version of the Hodgkin–Huxley model, which plays a significant role in the theory of nerve conduction. However, the majority of researchers considered the system of equations (1) under the assumption of $\epsilon_1, \epsilon_2 \neq 0$ [3, 4]. At the same time, cases of degenerate systems ($\epsilon_1 = 0$ or $\epsilon_2 = 0$) remained poorly understood, the necessity of studying of which is connected with the fact that the rate of change of one of the components of system (1) can significantly exceed another one. In case of $\epsilon_1 = 0$, the phase space of the system is a simple Banach C^∞-manifold, therefore, the problem has a unique solution. The question of the solvability of the Showalter–Sidorov–Dirichlet problem for Fitz Hugh–Nagumo system (1) in the case $\epsilon_1 = 0$ was considered in

papers [5,6], it was also studied their optimal control, start control and final observations for this system. In this article, we will be interested in case of $\epsilon_2 = 0$. In this case, the phase space of system of equations (1) contains singularity of Whitney fold type [7], which leads to nonuniqueness of solutions.

Consider degenerate system of equations (1) in case $\epsilon_2 = 0$ in cylinder $Q = \Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^∞:

$$\begin{cases} v_t = \alpha_1 v_{ss} + \beta_1 w - \kappa_1 v, \\ 0 = \alpha_2 w_{ss} + \beta_2 w - \kappa_2 v - w^3, \end{cases} \quad v = v(s,t), w = w(s,t), \tag{2}$$

with boundary value conditions

$$v(s,t) = 0,\ w(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbb{R}_+, \tag{3}$$

and initial value condition

$$v(0) = v_0. \tag{4}$$

Problem (2) – (4) can be investigated within the framework of abstract Showalter–Sidorov problem

$$L(u(0) - u_0) = 0 \tag{5}$$

for semilinear Sobolev type equation

$$L\dot{u} = Mu + N(u),\ \ker L \neq \{0\} \tag{6}$$

in specially constructed function spaces. Here $L \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$, N is nonlinear operator, $\mathfrak{U}, \mathfrak{F}$ are Banach spaces. By the phase space of equation (6) we mean the closure of the set of all admissible initial values, for which there is a local solution to problem (5), (6) [8]. So, based on the theory of (L,p)-bounded operators or (L,p)-sectorial operators, G.A. Sviridyuk, and later his adherers [9,10], found the conditions for the unique solvability of problem (5), (6). Namely, when the operator M is (L,p)-sectorial (bounded) and the phase space of equation (6) is a simple Banach C^∞-manifold, there is a single quasistationary the (semi)trajectory of problem (5), (6) passing through point u_0, which lies pointwise in phase space [11]. Recall that Banach C^∞-manifold is called simple if any of its atlas is equivalent to an atlas containing a single chart. In particular, if operator M is $(L,0)$-sectorial (bounded), then any solution (5), (6) will be a quasistationary (semi)trajectory. The main method for studying problem (2) – (4) is the phase space method. Following it, we construct set $\mathfrak{M} = \{u \in \mathfrak{U} : (I - Q)(Mu + N(u)) = 0\}$, then all solutions of problem (1), (3) lie in set $\mathfrak{M}$ as trajectories, where Q is spectral projector [11].

Back in 1987, G.A. Sviridyuk suggested that the solution to problem (5), (6) may not be unique if phase space of equation (6) is not simple Banach C^∞-manifold. In review [12] it was shown that initial value condition (5) for (6) can have several solutions in cases where phase space of (6) lies on smooth Banach manifold having singularities such as Whitney folds. For example, the Showalter–Sidorov problem for the Korpusov–Pletner–Sveshnikov equation may have two different solutions [13], and for the system of Plotnikov equations – three [14]. In work [7] it was shown that in degenerate case (for $\epsilon_2 = 0$) phase space of (2) contains singularity such as Whitney folds, therefore, it can have one or more solutions or the solution may not exist. In the course of this study, we will identify the conditions for the existence and uniqueness or multiplicity of solutions of Showalter–Sidorov problem (4) for Fitz Hugh–Nagumo system (2) depending on the parameters of the system.

1. The Morphology of Phase Space

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^∞. In cylinder $Q = \Omega \times \mathbb{R}_+$ we consider system of equations (2) with boundary value conditions (3) and initial value condition (4). We set $\mathfrak{H}_i = W_2^1(\Omega)$, $i = 1, 2$ and define space

$$\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2 = W_2^1(\Omega) \times W_2^1(\Omega).$$

Let vector functions $u = (v, w), \zeta = (\xi, \eta)$, consider Hilbert space $\mathcal{H} = L_2(\Omega) \times L_2(\Omega)$ with scalar product

$$[u, \zeta]_{\mathcal{H}} = \langle v, \xi \rangle_{\mathfrak{H}_1} + \langle w, \eta \rangle_{\mathfrak{H}_2},$$

and space $\mathfrak{U}_N = L_4(\Omega) \times L_4(\Omega)$. By $\mathfrak{U} = \mathfrak{F}$ we denote the space adjoint to $\mathfrak{H}$ with respect to duality of $[\cdot, \cdot]$ in $\mathcal{H}$. By virtue of the Sobolev embedding Theorems there are dense and continuous embeddings

$$\mathfrak{H} \hookrightarrow \mathfrak{U}_N \hookrightarrow \mathcal{H} \hookrightarrow \mathfrak{U}_N^* \hookrightarrow \mathfrak{U} = \mathfrak{F}. \tag{7}$$

Note that space $\mathcal{H}$ is identified with its adjoint. Construct linear operators $L, M : \mathfrak{U} \to \mathfrak{F}$

$$[Lu, \zeta] = \langle v, \xi \rangle,\ u, \zeta \in \mathfrak{U},$$

$$[Mu, \zeta] = -\alpha_1 \langle v_{s_i}, \xi_{s_i} \rangle - \alpha_2 \langle w_{s_i}, \eta_{s_i} \rangle,\ u, \zeta \in \mathfrak{U},\ \text{where } \operatorname{dom} M = \mathfrak{H}$$

and nonlinear operator

$$[N(u), \zeta] = \langle \beta_1 w - \kappa_1 v, \xi \rangle + \langle \beta_2 w - \kappa_2 v - w^3, \eta \rangle,\ \text{where } \operatorname{dom} N = \mathfrak{U}_N.$$

(Note that the Einstein agreement on summation over repeated indices is fulfilled everywhere.) By construction, operator $L \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$.

Denote by

$$\mathfrak{U}^0 = \ker L = \{0\} \times W_2^{-1}(\Omega),\ \mathfrak{U}^1 = W_2^{-1}(\Omega) \times \{0\},$$

$$\mathfrak{F}^1 = \operatorname{im} L = W_2^{-1}(\Omega) \times \{0\},\ \mathfrak{F}^0 = M[\mathfrak{U}^0 \cap \operatorname{dom} M] = \{0\} \times W_2^{-1}(\Omega),$$

when $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. Set L_1 as the restriction of operator L to $\mathfrak{U}^1$, then $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1, \mathfrak{U}^1)$.

Lemma 1. *For any $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\beta_1, \beta_2, \kappa_1, \kappa_2 \in \mathbb{R}$, $n \leq 4$*
(i) the operator M is $(L, 0)$-sectorial;
(ii) $N \in C^1(\mathfrak{U}_N; \mathfrak{U}_N^*)$.

Proof. (i) $(L, 0)$-sectoriality operator M was shown in work [14].
(ii) We show that $N \in C^1(\mathfrak{U}_N; \mathfrak{U}_N^*)$, where $\mathfrak{U}_N^*$ is dual space of $\mathfrak{U}_N$ with respect to duality of $[\cdot, \cdot]$. Indeed, due to Hölder inequality, we have

$$|[N(u), \zeta]| \leq (C_1 ||u||_{\mathfrak{U}_N}^3 + C_2 ||u||_{\mathfrak{U}_N}) ||\zeta||_{\mathfrak{U}_N},$$

$$|[N_u' \zeta_1, \zeta_2]| = \left| \int_\Omega (\beta_1 \xi_1 \xi_2 - \kappa_1 \xi_1 \xi_2) ds + \int_\Omega (\beta_2 \eta_1 \eta_2 - \kappa_2 \eta_1 \eta_2 - 3w^2 \eta_1 \eta_2) ds \right| \leq$$
$$\leq (C_3 ||u||_{\mathfrak{U}_N}^2 + C_4) \cdot ||\zeta_1||_{\mathfrak{U}_N} \cdot ||\zeta_2||_{\mathfrak{U}_N},$$

where constants $C_i \in \mathbb{R}_+, i = \overline{1, 4}$, depend neither on u, nor on ζ, ζ_1, ζ_2. Here N_u' is the Frechet derivative of operator N at point u. The inclusion of $N \in C^1(\mathfrak{U}_N; \mathfrak{U}_N^*)$ is proved. □

Thus, we reduced problem (2), (3) to a semilinear equation of Sobolev type (6). Note that condition (4) takes form (5). We are interested in the solvability of problem (2) – (4) for any $u_0 = (v_0, w_0) \in \mathfrak{H}$.

Let $\{\nu_k\}$ denote the sequence of eigenvalues of the following spectral problem:

$$\begin{gathered} -\Delta\varphi = \nu\varphi,\ s \in \Omega, \\ \varphi(s) = 0,\ s \in \partial\Omega, \end{gathered} \tag{8}$$

where eigenvalues are numbered in nondecreasing order of their multiplicity. Denote by $\{\varphi_k\}$ the corresponding eigenfunctions orthonormal in the sense of scalar product $\langle\cdot,\cdot\rangle$ in $L_2(\Omega)$.

Definition 1. *Vector-function $u \in C^1((0,\tau);\mathfrak{U}) \cap C((0,\tau);\mathfrak{U}_N)$, satisfying equation (2), is called the solution of the equation. Solution $u = u(t)$ of equation (2) is called the solution of problem (2), (4) if*

$$\lim_{t\to 0+} ||L(u(t) - u_0)||_{\mathfrak{F}} = 0.$$

Build

$$\mathfrak{M} = \left\{ u \in \mathfrak{H} : -\langle v, \eta\rangle = \left\langle -\frac{\beta_2}{\kappa_2} w + \frac{1}{\kappa_2} w^3, \eta \right\rangle + \left\langle \frac{\alpha_2}{\kappa_2} w_{s_i}, \eta_{s_i} \right\rangle \right\}$$

and note that all solutions of system of equations (2) satisfying boundary value conditions (3) will lie in this set.

Lemma 2. *Let $\alpha_2, \kappa_2 \in \mathbb{R}_+$, $\beta_2 \in (0, \alpha_2\nu_1)$, $n \le 4$, then for any vector $v \in \mathfrak{H}_1$ there exists unique vector $w \in \mathfrak{H}_2$ such that $u = col(v, w) \in \mathfrak{M}$.*

Proof. Construct an auxiliary operator

$$\langle A(w), \eta\rangle = \left\langle -\frac{\beta_2}{\kappa_2} w + \frac{1}{\kappa_2} w^3, \eta \right\rangle + \left\langle \frac{\alpha_2}{\kappa_2} w_{s_i}, \eta_{s_i} \right\rangle,\ w, \eta \in \mathfrak{H}_2,\ \operatorname{dom} A = \mathfrak{H}_2.$$

Denote by $\mathfrak{H}_2^*$ the space conjugate to $\mathfrak{H}_2$ with respect to duality of $\langle\cdot,\cdot\rangle$. Insofar as

$$|\langle A(w_1), w_2\rangle| \le C_1(||w_1||_{\mathfrak{H}_2} + ||w_1||_{\mathfrak{H}_2}^3)||w_2||_{\mathfrak{H}_2},$$

where constant $C_1 \in \mathbb{R}_+$ depends on $\beta_2, \kappa_2, \alpha_2$ and embedding constants (7) and does not depend on w, thus the action of operator $A : \mathfrak{H}_2 \to \mathfrak{H}_2^*$ is proved. Note that operator $A : \mathfrak{H}_2 \to \mathfrak{H}_2^*$ is coercive, i.e.

$$\lim_{||w||_{\mathfrak{H}_2}\to+\infty} \langle A(w), w\rangle ||w||_{\mathfrak{H}_2}^{-1} = \lim_{||w||_{\mathfrak{H}_2}\to+\infty} \left(\int_\Omega (-\tfrac{\beta_2}{\kappa_2} w^2 - \tfrac{\alpha_2}{\kappa_2}(w_{s_i})^2 + \tfrac{1}{\kappa_2} w^4) ds \right) \cdot \left(\int_\Omega (w_{s_i}^2 + w^2) ds \right)^{-\frac{1}{2}} = +\infty.$$

In addition, operator A is strictly monotone, that is,

$$\langle A(w_1) - A(w_2), w_1 - w_2\rangle = \int_\Omega (-\tfrac{\beta_2}{\kappa_2}(w_1 - w_2)^2 - \tfrac{\alpha_2}{\kappa_2}(w_{1_{s_i}} - w_{2_{s_i}})^2 + \tfrac{1}{\kappa_2} \cdot (w_1 - w_2)^2 (w_1^2 + w_1 w_2 + w_2^2)) ds > 0 \,\forall w_1,\ w_2 \in \mathfrak{H}_2,$$

as soon as $w_1 \ne w_2$. Finally, we show the smoothness of operator A. Indeed,

$$|\langle A'_w\xi,\eta\rangle| = |\int\limits_\Omega(-\frac{\beta_2}{\kappa_2}\eta\xi - \frac{\alpha_2}{\kappa_2}\eta_{s_i}\xi_{s_i} + \frac{1}{\kappa_2}3w^2\eta\xi)ds| \le (C_1 + C_2||w||^2_{\mathfrak{H}_2})||\eta||_{\mathfrak{H}_2}||\xi||_{\mathfrak{H}_2},$$

where constants C_1, C_2 depend only on $\alpha_2, \beta_2, \kappa_2$ and the nesting constants. Hence, by virtue of the Vishik–Minty–Browder Theorem [15], equation $A(w) = -v$ has a unique solution.

□

Consider the case of $\beta_2 = \alpha_2\nu_1$, put

$$\mathfrak{H}_1^\perp = \{v^\perp \in \mathfrak{H}_1 : \langle v^\perp, \varphi\rangle = 0\},\ \mathfrak{H}_2^\perp = \{w^\perp \in \mathfrak{H}_2 : \langle w^\perp, \varphi\rangle = 0\}.$$

Let ν_1 be a single root and φ be an eigenfunction of problem (8), corresponding to the eigenvalue of ν_1, normalized in sense $L_2(\Omega)$. If $v \in \mathfrak{H}_1$ and $w \in \mathfrak{H}_2$ be represented as $v = v^\perp + r\varphi$ and $w = w^\perp + q\varphi$, where $r, q \in \mathbb{R}$, then set $\mathfrak{M}$ takes the following form:

$$\mathfrak{M} = \left\{ u \in \mathfrak{H} : \left\{ \begin{array}{l} -v^\perp = -\frac{\beta_2}{\kappa_2}w^\perp - \frac{\alpha_2}{\kappa_2}\Delta w^\perp + \frac{1}{\kappa_2}\int\limits_\Omega (w^\perp + q\varphi)^3\varphi ds, \\ -\kappa_2 r = \int\limits_\Omega (w^\perp + q\varphi)^3\varphi ds. \end{array} \right\}. \right. \tag{9}$$

Lemma 3. *Let $\alpha_2, \kappa_2 \in \mathbb{R}_+$, $\beta_2 = \alpha_2\nu_1$, $n \le 4$, then for any vector $v^\perp \in \mathfrak{H}_1^\perp$ there exists unique vector $w^\perp \in \mathfrak{H}_2^\perp$ such that*

$$v^\perp = \frac{\beta_2}{\kappa_2}w^\perp + \frac{\alpha_2}{\kappa_2}\Delta w^\perp - \frac{1}{\kappa_2}\int\limits_\Omega (w^\perp + q\varphi)^3\varphi ds.$$

The proof of this lemma is carried out similarly to the proof of Lemma 2, if we consider the following operator as an auxiliary operator:

$$A(w^\perp) = \frac{-\beta_2}{\kappa_2}w^\perp - \frac{\alpha_2}{\kappa_2}\Delta w^\perp + \frac{1}{\kappa_2}\int\limits_\Omega (w^\perp + q\varphi)^3\varphi ds.$$

By Lemma 3 by $v_0^\perp$ and r_0, we construct $w_0^\perp$ and q_0. Put $v_0 = v_0^\perp + r_0\varphi$ and $w_0 = w_0^\perp + q_0\varphi$, then $u_0 = (v_0, w_0) \in \mathfrak{M}$.

Theorem 1. *Let $\alpha_2, \kappa_2 \in \mathbb{R}_+$, $n \le 4$, $\beta_2 \in (0, \alpha_2\nu_1)$, or $\beta_2 = \alpha_2\nu_1$, $q^2||\varphi||^4_{L_4} + 2q\int\limits_\Omega w^\perp\varphi^3 ds + \int\limits_\Omega (w^\perp)^2\varphi^2 \ne 0$, then the set $\mathfrak{M}$ at the point u_0 is a simple Banach C^∞-manifold.*

The second equation of system (9) can be represented as:

$$q^3||\varphi||^4_{L_4(\Omega)} + 3q^2\int\limits_\Omega w^\perp\varphi^3 ds + 3q\int\limits_\Omega (w^\perp)^2\varphi^2 ds + \int\limits_\Omega \varphi(w^\perp)^3 ds + \kappa_2 r = 0. \tag{10}$$

The equation (10) is a cubic equation of general form $aq^3 + bq^2 + cq^2 + d = 0$ with respect to q. According to Cardano formulas, any cubic equation of general form with the help of replacement $q = y - \frac{b}{3a}$ can be reduced to canonical form $y^3 + py + e = 0$ with coefficients

$$a = ||\varphi||^4_{L_4(\Omega)},\ b = 3\int\limits_\Omega w^\perp\varphi^3 ds,\ c = 3\int\limits_\Omega (w^\perp)^2\varphi^2 ds,\ d = \int\limits_\Omega \varphi(w^\perp)^3 ds - \kappa_2 r,$$

$$p = \frac{3ac - b^2}{9a^2},\ e = \frac{1}{2}\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right),\ Q = p^3 + e^2.$$

By virtue of the already mentioned Cardano formulas, Theorem 1 and Theorem on the existence of a solution of problem (5), (6) [8, 10] is valid.

Theorem 2. *For any* $u_0 = (v_0, w_0) \in \mathfrak{H}$, $n \leq 4$, $\alpha_2, \kappa_2 \in \mathbb{R}_+$ *and*
(i) $\beta_2 \in (0, \alpha_2\nu_1)$ *there exists a unique solution to problem (2) – (4);*
(ii) $\beta_2 = \alpha_2\nu_1$, $Q > 0$ *there exists a unique solution to problem (2) – (4);*
(iii) $\beta_2 = \alpha_2\nu_1$, $Q = 0$ *and following condition is fulfilled*

$$q^2||\varphi||^4_{L_4(\Omega)} + 2q\int\limits_\Omega w^\perp\varphi^3 ds + \int\limits_\Omega (w^\perp)^2\varphi^2 = 0$$

there exists two solutions to problem (2) – (4);
(iv) $\beta_2 = \alpha_2\nu_1$, $Q < 0$ *there exists three solutions to problem (2) – (4).*

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О НЕЕДИНСТВЕННОСТИ РЕШЕНИЙ ЗАДАЧИ ШОУОЛТЕРА – СИДОРОВА ДЛЯ ОДНОЙ МАТЕМАТИЧЕСКОЙ МОДЕЛИ РАСПРОСТРАНЕНИЯ НЕРВНОГО ИМПУЛЬСА В МЕМБРАНЕ

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Статья посвящена изучению морфологии фазового пространства математической модели распространения нервного импульса в мембране, основанной на вырожденной системе уравнений Фитц Хью – Нагумо, заданной на ограниченной области с гладкой границей. В данной математической модели скорость изменения одной из компонент системы может значительно превосходить другую, что приводит к вырожденной системе уравнений Фитц Хью – Нагумо. Изучаемая модель относится к широкому классу полулинейных моделей соболевского типа. Для исследования вопроса неединственности решений задачи Шоуолтера – Сидорова будет использован метод фазового пространства, который был разработан Г.А. Свиридюком для исследования разрешимости уравнений соболевского типа. Нами будет показано, что фазовое пространство исследуемой модели содержит особенности типа складки Уитни и выявлены условия существования, единственности или множественности решений задачи Шоуолтера – Сидорова в зависимости от параметров системы.

Ключевые слова: уравнения соболевского типа; задача Шоуолтера – Сидорова; система уравнений Фитц Хью – Нагумо; неединственность решений.

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