КРАТКИЕ СООБЩЕНИЯ

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THE BARENBLATT–ZHELTOV–KOCHINA MODEL ON THE SEGMENT WITH WENTZELL BOUNDARY CONDITIONS

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In terms of the theory of relative p-bounded operators, we study the Barenblatt–Zheltov–Kochina model, which describes dynamics of pressure of a filtered fluid in a fractured-porous medium with general Wentzell boundary conditions. In particular, we consider spectrum of one-dimensional Laplace operator on the segment $[0,1]$ with general Wentzell boundary conditions. We examine the relative spectrum in one-dimensional Barenblatt–Zheltov–Kochina equation, and construct the resolving group in the Cauchy-Wentzell problem with general Wentzell boundary conditions. In the paper, these problems are solved under the assumption that the initial space is a contraction of the space $L^2(0,1)$.

Keywords: Barenblatt–Zheltov–Kochina model; relatively p-bounded operator; phase space; C_0-contraction semigroups; Wentzell boundary conditions.

Dedicated to the 60-th birthday of outstanding mathematician Jacek Banasiak.

Introduction

Let us consider the Cauchy–Wentzell problem

$$\begin{aligned} &u(x,0) = v_0(x),\ x \in [0,1],\\ &u_{xx}(0,t) + \alpha_0 u_x(0,t) + \alpha_1 u(0,t) = 0,\\ &u_{xx}(1,t) + \beta_0 u_x(1,t) + \beta_1 u(1,t) = 0 \end{aligned} \tag{1}$$

for the Barenblatt–Zheltov–Kochina equation on the segment $[0,1]$

$$\lambda u_t(x,t) - u_{txx}(x,t) = \alpha u_{xx}(x,t) + f(x,t), \quad (x,t) \in [0,1] \times \mathbb{R}_+, \tag{2}$$

which describes dynamics of pressure of a filtered fluid in a fractured-porous medium. Here α and λ are the material parameters characterizing the environment, the parameter $\alpha \in \mathbb{R}_+$, the function $f = f(x,t)$ plays the role of external loading.

For the first time Wentzell boundary condition were considered in [1] in order to find diffusive processes for Markov processes homogeneous in time on the segment. Independently, these conditions were investigated in [2]. More general case were studied later in [3]. Namely, the domain belongs to n-dimensional Euclidean space which is a circle or a sphere, and semigroup is C_0-contracting and invariant under rotations.

Further the results of [3] were developed and generalized in papers [4–7]. In particular, the classification of general Wentzell boundary conditions for a fourth-order differential operator in the one-dimensional case was established in [4], the role of Wentzell boundary conditions in linear and nonlinear analysis was shown in [5], Wentzell boundary conditions for the Sturm–Liouville operator were studied in [6], the Laplace operator with general Wentzell boundary condition in Sobolev space was considered in [7]. These papers formed the basis of the new scientific direction, which endures a blossoming time in the present.

The purpose of this work is to research resolvability of problem (1) – (2) with Wentzell boundary conditions. The article contains two sections except introduction, conclusion, and references. The relative spectrum of the Laplace operator with Wentzell boundary condition is found in the first section. The main results on resolvability of the Cauchy–Wentzell problem in the Barenblatt–Zheltova–Kochina model are given in the second section.

1. Relative Spectrum of the Laplace Operator with Wentzell Boundary Condition

Let us consider the differential operator

$$Au(x) = u''(x), \quad x \in [0,1] \tag{3}$$

with general Wentzell boundary conditions

$$Au(0) + \alpha_0 u'(0) + \alpha_1 u(0) = 0, \tag{4}$$

$$Au(1) + \beta_0 u'(1) + \beta_1 u(1) = 0. \tag{5}$$

By formulas (3) – (5) we define the linear operator $A : dom\ A \subset \mathfrak{F} \to \mathfrak{F}$. Here $\mathfrak{F}$ is a space $\left(L^2[0,1], dx\Big|_{(0,1)} \oplus \eta ds\Big|_{\{0,1\}} \right)$ with the norm

$$\|u\|^2_{\mathfrak{F}} = \int\limits_0^1 |u(x)|^2 dx + \eta_0 |u(0)|^2 + \eta_1 |u(1)|,$$

where dx is a Lebesque measure on the segment $(0,1)$, ds is a point measure at the boundary, $\eta_0 = \frac{1}{-\alpha_1}$, $\eta_1 = \frac{1}{\beta_1}$, where $\alpha_1 < 0 < \beta_1$, are positive weights. The full construction of the space $\mathfrak{F}$ is given in [8]. We consider also the linear manifold $dom\ A = \{u \in C^2[0,1] : \text{conditions } (4), (5) \text{ are fulfilled}\}$ as the domain of the operator A.

Lemma 1. *Let the operator A be defined by formulas* (3)–(5)*. Then*

(i) $dom\ A = \{u \in C^2[0,1]$: conditions (4), (5) are fulfilled$\}$ is a Banach space with regard to the norm $\|u\|_{C^2[0,1]}$;

(ii) dom A is densely embedded in $\mathfrak{F}$;

(iii) $A \in \mathcal{L}(\text{dom } A; \mathfrak{F})$.

Let us give an idea of the proof. Statement *(i)* is obviously, since dom A forms a subspace closed in $C^2[0,1]$. Statement *(ii)* obviuosly follows from the fact that the operator of embedding $\mathcal{G} : C^2[0,1] \to \mathfrak{F}$ is compact. Statement *(iii)* is obviously.

We consider the spectral problem for the operator A with general Wentzell boundary conditions. Prove the following theorem.

Theorem 1. *Suppose that the operator A satisfies the conditions of Lemma 1. Then A has a real, discrete, finite multiplicity spectrum with the unique limit point at infinity.*

Proof. It follows from [8] that the operator A on $\mathfrak{F}$ is essentially self-adjoint. This means that the spectrum of the operator A is real. Let us define the spectrum of the operator A and find its resolvent. We have $(\lambda\mathbb{I} - A)u = f(x),\ x \in [0,1],$ for $f \in C^2[0,1]$.

Consider the case of $\lambda < 0$. Solve the differential equation with general Wentzell boundary conditions by classical methods, and obtain the resolvent of the following form:

$$u(x) = (\lambda\mathbb{I} - A)^{-1} f = R_\lambda f = \overline{C_1}\cos(\sqrt{-\lambda}x) + \overline{C_2}\sin(\sqrt{-\lambda}x) + \int\limits_0^x \frac{f(s)}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}(x-s))ds.$$

Write coefficients $\overline{C_1} = \dfrac{A_0}{B}$ and $\overline{C_2} = \dfrac{A_1}{B}$ for the resolvent, where

$$B = (\lambda + \alpha_1)\Big(\lambda\sin(\sqrt{-\lambda}) + \beta_0\sqrt{-\lambda}\cos(\sqrt{-\lambda}) + \beta_1\sin(\sqrt{-\lambda})\Big) - \alpha_0\sqrt{-\lambda}\Big(\lambda\cos(\sqrt{-\lambda}) -$$
$$-\beta_0\sqrt{-\lambda}\sin(\sqrt{-\lambda}) + \beta_1\cos(\sqrt{-\lambda})\Big),$$

$$A_0 = f(0)\Big(\lambda\sin(\sqrt{-\lambda}) + \beta_0\sqrt{-\lambda}\cos(\sqrt{-\lambda}) + \beta_1\sin(\sqrt{-\lambda})\Big) - \alpha_0\sqrt{-\lambda}\Big(f(1) -$$
$$-\int\limits_0^1 \tfrac{f(s)\lambda}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}(1-s))ds - \beta_0\int\limits_0^1 f(s)\cos(\sqrt{-\lambda}(1-s))ds - \beta_1\int\limits_0^1 \tfrac{f(s)}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}(1-s))ds\Big),$$

$$A_1 = (\lambda + \alpha_1)\Big(f(1) - \int\limits_0^1 \tfrac{f(s)\lambda}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}(1-s))ds - \beta_0\int\limits_0^1 f(s)\cos(\sqrt{-\lambda}(1-s))ds -$$
$$-\beta_1\int\limits_0^1 \tfrac{f(s)}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}(1-s))ds\Big) - f(0)\Big(\lambda\cos(\sqrt{-\lambda}) - \beta_0\sqrt{-\lambda}\sin(\sqrt{-\lambda}) + \beta_1\cos(\sqrt{-\lambda})\Big).$$

The resolvent operator R_λ is the sum of a two-dimensional operator (a linear combination of sine and cosine) and an integral operator of Hilbert–Schmidt type. A two-dimensional operator is finite-dimensional, and hence compact, since the coefficients $\overline{C_1}$ and $\overline{C_2}$ depend continuously on f in the metric of $\mathfrak{F}$. Hence, the operator $R_\lambda = (\lambda\mathbb{I} - A)^{-1}$ is compact in $\mathfrak{F}$ as the sum of finite-dimensional and compact operators. By Hilbert's theorem, R_λ has a discrete, finite multiplicity spectrum with the unique limit point at zero.

Let us show that the operator A has a discrete, finite multiplicity spectrum with the unique limit point at infinity. Fix an arbitrary eigenvalue λ_0 of the operator R_λ and express the eigenvalues of the operator A through the eigenvalues of the resolvent R_λ. We obtain $R_\lambda f = \lambda_0 f$, where f is the eigenvector of the resolvent. By acting with the operator $(\lambda\mathbb{I} - A)$ on both parts of the equality and dividing by λ_0 ($\lambda_0 \neq 0$), we get the expression

$$Af = \left(\lambda\mathbb{I} - \frac{1}{\lambda_0}\right)f,$$

which shows how the eigenvalues of the original and resolvent operators are related. Due to the behavior of the spectrum of the operator R_λ, we proved that for $\lambda < 0$ the operator A has a discrete, finite multiplicity spectrum with the unique limit point at infinity.

Similarly, consideration of the case $\lambda > 0$ by the Sturm–Liouville method shows that the set of eigenvalues is finite or empty depending on the conditions on the coefficients in (4), (5).

Consider the case of $\lambda = 0$. Find sufficient conditions for the set of eigenvalues of the operator A. Note that if the coefficients in (4), (5) satisfy the equality

$$\alpha_0\beta_1 = \alpha_1(\beta_0 + \beta_1),$$

then $\lambda = 0$ belongs to the set of eigenvalues of the operator A. The theorem is proved. $\square$

The Barenblatt–Zheltov–Kochina equation

$$\lambda u_t(x,t) - u_{txx}(x,t) = \alpha u_{xx}(x,t) + f(x,t), \quad (x,t) \in [0,1] \times \mathbb{R}_+$$

can be considered as a non-homogeneous Sobolev type equation $Lu_t = Mu + f$, where the operators $L = \lambda - A \in \mathcal{L}(\mathrm{dom}A; \mathfrak{F})$, $M = \alpha A \in \mathcal{L}(\mathrm{dom}A; \mathfrak{F})$, the function $f = f(x,t) \in C^2([0,1] \times \mathbb{R}_+; \mathfrak{F})$. In order to solve the Cauchy–Wentzell problem (6), (7), we find the *L-spectrum* operator of M. Since the *L-resolvent* of the operator M takes the form

$$(\mu L - M)^{-1} = (\mu(\lambda - A) - \alpha A)^{-1} = \{\mu + \alpha \neq 0\} = (\mu + \alpha)^{-1} \left[\frac{\mu \lambda}{\mu + \alpha} - A \right]^{-1}$$

with $\mu + \alpha \neq 0$, then μ belongs to relative spectrum $\sigma^L(M)$ if

$$\mu = \frac{\alpha \sigma(A)}{\lambda - \sigma(A)}.$$

Therefore, according to Theorem 1, with $\mu + \alpha \neq 0$, we have a discrete, finite *L-spectrum* $\sigma^L(M)$ of the operator M with the limit point $-\alpha$ at infinity.

Consider the case of $\mu + \alpha = 0$. With $\lambda = 0$ we have $\sigma^L(M) = \{-\alpha\}$. With $\lambda \neq 0$ we have $\sigma^L(M) = \{\varnothing\}$, if $\alpha \neq 0$, and $\sigma^L(M) = \{0\}$, if $\alpha = 0$. We described the *L-spectrum* of the operator M, getting the following corollary of Theorem 1.

Corollary 1. *The L-spectrum of the operator M in the Barenblatt–Zheltov–Kochina equation with Wentzell boundary conditions is discrete, finite multiplicity, with the limit point $-\alpha$ at infinity.*

2. The Cauchy–Wentzell Problem in the Barenblatt–Zheltov–Kochina Model

Let us consider the Cauchy–Wentzel problem in the previously introduced space $\mathfrak{F}$ on the segment $[0,1]$

$$\begin{aligned} &u(x,0) = v_0(x), \ x \in [0,1], \\ &u_{xx}(0,t) + \alpha_0 u_x(0,t) + \alpha_1 u(0,t) = 0, \\ &u_{xx}(1,t) + \beta_0 u_x(1,t) + \beta_1 u(1,t) = 0 \end{aligned} \tag{6}$$

for the Barenblatt–Zheltov–Kochina equation

$$\lambda u_t(x,t) - u_{txx}(x,t) = \alpha u_{xx}(x,t) + f(x,t), \quad (x,t) \in [0,1] \times \mathbb{R}_+. \tag{7}$$

By Corollary 1, the operator M is (L,σ)-bounded, therefore, the following theorem holds.

Theorem 2. *Suppose that the linear operator A satisfies the conditions of Lemma 1, and $f \in \mathfrak{F}$ is a fixed vector. Then*

(i) if $\lambda \notin \sigma(A)$, then for any $v_0 \in domA$ and $f \in \mathfrak{F}$ there exists the unique solution $u \in C^2(\mathbb{R}; domA)$ to problem (6)–(7)*, which has the form*

$$u(x,t) = \sum_{k=1}^{\infty} e^{\frac{\alpha\lambda_k}{\lambda - \lambda_k} t} < v_0, \varphi_k >_{\mathfrak{F}} \varphi_k(x) + \sum_{k=1}^{\infty} \left(e^{\frac{\alpha\lambda_k}{\lambda - \lambda_k} t} - 1 \right) \frac{< f, \varphi_k >_{\mathfrak{F}}}{\alpha \lambda_k} \varphi_k(x);$$

(ii) if $\lambda \in \sigma(A)$, then for any $f \in \mathfrak{F}$ and $v_0 \in \mathfrak{P}_f = \Big\{ u \in domA : \alpha\lambda < u, \varphi_k >_{\mathfrak{F}} = - < f, \varphi_k >_{\mathfrak{F}}, \lambda_k = \lambda \Big\}$ there exists the unique solution $u \in C^2(\mathbb{R}; \mathfrak{P}_f)$ to problem (6), (7)*, which has the form*

$$u(x,t) = -\frac{1}{\alpha\lambda}\sum_{\lambda=\lambda_k} < f,\varphi_k >_{\mathfrak{F}} \varphi_k(x) + \sum_{\lambda\neq\lambda_k} e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t} < v_0(x),\varphi_k >_{\mathfrak{F}} \varphi_k(x) +$$
$$+\sum_{\lambda\neq\lambda_k}\left(e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t} - 1\right)\frac{< f,\varphi_k >_{\mathfrak{F}}}{\alpha\lambda_k}\varphi_k(x).$$

Proof. The proof of this theorem depends on the kernel of the operator L and consists in applying either the classical theorem for a non-homogeneous differential operator equation, or Sviridyuk's theorem. According to Theorem 1, the Laplace operator has a real, discrete, finite multiplicity spectrum having the limit point at $-\infty$, and $\{\lambda_k : k \in \mathbb{N}\}$ are eigenvalues of the Laplace operator, which are numbered in non-increasing order taking into account the multiplicity, and correspond to eigenfunctions $\{\varphi_k : k \in \mathbb{N}\}$. Then, according to the completeness of the eigenfunctions, for $v \in \mathfrak{F}$ we have

$$R_\mu(A)v = (\mu\mathbb{I} - \Delta)^{-1}v = \sum_{k=1}^{\infty}\frac{< v,\varphi_k >_{\mathfrak{F}} \varphi_k}{\mu - \lambda_k},$$

and, therefore,

$$R^L_\mu(M)v = (\mu L - M)^{-1} = \sum_{k=1}^{\infty}\frac{< v,\varphi_k >_{\mathfrak{F}} (\lambda - \lambda_k)}{\mu(\lambda-\lambda_k) - \alpha\lambda_k}\varphi_k. \tag{8}$$

Termwise integration is admissible, since the series uniform convergences by the norm of the space dom$A$. Therefore, substituting *L-resolvent* (8) of the operator M and applying the residue theorem, we obtain corresponding expressions *(i)*, *(ii)*. □

Conclusion

We constructed the resolution group in the Cauchy–Wentzell problem. To this end, we used the Sviridyuk's theory, and the space, the structure of which is specified in [8]. Further, we plan to continue the results of the paper by applying the Wentzell boundary conditions in directions related to [10, 11].

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МОДЕЛЬ БАРЕНБЛАТТА – ЖЕЛТОВА – КОЧИНОЙ В ОБЛАСТИ С ГРАНИЧНЫМИ УСЛОВИЯМИ ВЕНТЦЕЛЯ

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В терминах теории относительно p-ограниченных операторов исследуется модель Баренблатта – Желтова – Кочиной, описывающая динамику давления фильтрующейся жидкости в трещинновато-пористой среде с общими граничными условиями Вентцеля. В частности, рассматривается спектр одномерного оператора Лапласа на отрезке $[0,1]$ с общими граничными условиями Вентцеля; ставится вопрос об относительном спектре в одномерном уравнении Баренблатта – Желтова – Кочиной и построении разрешающей группы в задаче Коши – Вентцеля с общими граничными условиями Вентцеля. В работе решены указанные задачи в предположении, что исходное пространство, в котором действует оператор Лапласа на отрезке, есть сужение пространства $L^2(0,1)$.

Ключевые слова: модель Баренблатта – Желтова – Кочиной; относительно p-ограниченный оператор; фазовое пространство; C_0-сжимающие полугруппы; краевые условия Вентцеля.

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