

INVERSE PROBLEM FOR SOBOLEV TYPE MATHEMATICAL MODELS

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The work is devoted to the study of an inverse problem for the linear Sobolev type equation of higher order with an unknown coefficient depending on time. Since the equation might be degenerate the phase space method is used. It consists in construction of projectors splitting initial spaces into a direct sum of subspaces. Actions of operators also split. Therefore, the initial problem is reduced to two problems: regular and singular. The regular one is reduced to the first order nondegenerate problem which is solved via approximations. The needed smoothness of the solution is obtained. Then it is substituted into the singular problem which is solved using the methods of relatively polynomially bounded operator pencils theory. The main result of the work contains sufficient conditions for the existence and uniqueness of the solution to the inverse problem for a complete Sobolev type model of the second order. This technique can be used to investigate inverse problems of the considered type for Boussinesq–Love mathematical model.

Keywords: Sobolev type equation; inverse problem; mathematical models; equation of second order.

*Dedicated to the 60-th birthday
of outstanding mathematician Jacek Banasiak.*

Introduction

Let $\mathcal{U}, \mathcal{F}, \mathcal{Y}$ be Banach spaces, operators $A \in \mathcal{L}(\mathcal{U}; \mathcal{F})$, $\ker A \neq \{0\}$, $B_1, B_0 \in \mathcal{C}l(\mathcal{U}; \mathcal{F})$, $C \in \mathcal{L}(\mathcal{U}; \mathcal{Y})$, the functions $\chi : [0, T] \rightarrow \mathcal{L}(\mathcal{Y}; \mathcal{F})$, $f : [0, T] \rightarrow \mathcal{F}$, $\Psi : [0, T] \rightarrow \mathcal{Y}$. Consider the following problem

$$Av''(t) = B_1v'(t) + B_0v(t) + \chi(t)q(t) + f(t), \quad t \in [0, T], \quad (1)$$

$$v(0) = v_0, \quad (2)$$

$$v'(0) = v_1, \quad (3)$$

$$Cv(t) = \Psi(t). \quad (4)$$

One of the prototypes of equation (1) is

$$(\lambda - \Delta)v_{tt} = \alpha(\Delta - \lambda')v_t + \beta(\Delta - \lambda'')v + g,$$

which models, among others, the incompressible fluid free surface perturbation under the assumption of motion potentiality and conservation of mass in a layer [1], longitudinal vibrations of an elastic rod [2] and wave processes in smectic and plasma [3]. The problem

of finding a pair of functions $v(t) \in C^2([0, T]; \mathcal{U})$ and $q(t) \in C^1([0, T]; \mathcal{Y})$ from relations (1) – (4) is called the inverse problem.

Equation (1) refers to Sobolev type equations, since the operator A can be degenerate. Equations of this type of the first order are well studied. For example, there were proved the existence and uniqueness of classical solutions in the case of relatively sectorial operators [4], and [5] presented sufficient conditions for the unique solvability of the inverse problem using semigroups theory. In addition, there are articles devoted to the study of initial-final problems [6, 7], as well as to a numerical study of equations on the graphs [8], where the solution is sought in the form of the Galerkin sum. Sobolev type equations find their application in solving various technical problems, such as in [9]. Higher order equations were studied in [10] and [11] in the case of relatively bounded operator pencils. Semigroups theory finds application also in mathematical bioscience [12].

Problem (1) – (4) is inverse since, in addition to finding the function $v(t)$, we need to restore the function $q(t)$. Inverse problems were studied for the heat equation [13], for the second-order hyperbolic equations [14, 15], and also for the multidimensional wave equation of the second order [16].

1. Preliminary Information and Reduction of Initial Inverse Problem

To find a pair of functions $v(t)$ and $q(t)$ we use the results obtained in the research of higher order Sobolev type equations [17]. Denote by \vec{B} the pencil of operators B_1, B_0 .

Definition 1. *The pencil \vec{B} is called polynomially A -bounded if*

$$\exists a \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (R_\mu^A(\vec{B}) \in \mathcal{L}(\mathcal{F}; \mathcal{U})).$$

Introduce an important condition:

$$\int_{\gamma} R_\mu^A(\vec{B}) d\mu \equiv \mathbb{O}, \tag{A}$$

where $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Lemma 1. *Let the pencil \vec{B} be polynomially A -bounded and condition (A) be fulfilled. Then the operators*

$$P = \frac{1}{2\pi i} \int_{\gamma} R_\mu^A(\vec{B}) \mu A d\mu \in \mathcal{L}(\mathcal{U}), \quad Q = \frac{1}{2\pi i} \int_{\gamma} \mu A R_\mu^A(\vec{B}) d\mu \in \mathcal{L}(\mathcal{F})$$

are projectors.

Put $\mathcal{U}^0 = \ker P, \mathcal{F}^0 = \ker Q, \mathcal{U}^1 = \text{im } P, \mathcal{F}^1 = \text{im } Q$. From the previous Lemma it follows that $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1, \mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$. Let $A^k(B_l^k)$ denote the restriction of the operator $A(B_l)$ onto $\mathcal{U}^k, k = 0, 1; l = 0, 1$.

Theorem 1. *Let the pencil \vec{B} be polynomially A -bounded and condition (A) be fulfilled. Then the actions of the operators split:*

- (i) $A^k \in \mathcal{L}(\mathcal{U}^k; \mathcal{F}^k)$, $k = 0, 1$;
- (ii) $B_l^k \in \mathcal{L}(\mathcal{U}^k; \mathcal{F}^k)$, $k = 0, 1$, $l = 0, 1$;
- (iii) there exists an operator $(A^1)^{-1} \in \mathcal{L}(\mathcal{F}^1; \mathcal{U}^1)$;
- (iv) there exists an operator $(B_0^0)^{-1} \in \mathcal{L}(\mathcal{F}^0; \mathcal{U}^0)$.

Definition 2. Define the family of operators $\{K_q^1, K_q^2\}$ as follows:

$$\begin{aligned} K_1^1 &= H_0, \quad K_1^2 = -H_1, \\ K_{q+1}^1 &= K_q^2 H_0, \quad K_{q+1}^2 = K_q^1 - K_q^2 H_1, \quad q = 1, 2, \dots \end{aligned}$$

where $H_0 = (B_0^0)^{-1} A^0$; $H_1 = (B_0^0)^{-1} B_1^0$.

Definition 3. The point ∞ is called

- (i) a removable singular point of the A -resolvent of pencil \vec{B} , if $K_1^1 \equiv \mathbb{O}$, $K_1^2 \equiv \mathbb{O}$;
- (ii) a pole of order $p \in \mathbb{N}$ of the A -resolvent of pencil \vec{B} , if $K_p^1 \not\equiv \mathbb{O}$ or $K_p^2 \not\equiv \mathbb{O}$, but $K_{p+1}^1 \equiv \mathbb{O}$, $K_{p+1}^2 \equiv \mathbb{O}$;
- (iii) an essentially singular point of the A -resolvent of the pencil \vec{B} , if $K_k^2 \not\equiv \mathbb{O}$ for any $k \in \mathbb{N}$.

Let the pencil \vec{B} be polynomially A -bounded and condition (A) be fulfilled, then $v(t)$ can be represented as $v(t) = Pv(t) + (I - P)v(t) = u(t) + \omega(t)$. Suppose that $\mathcal{U}^0 \subset \ker C$. Then, by virtue of Theorem 1 and Lemma 1, problem (1) – (4) is equivalent to the problem of finding the functions $u \in C^2([0, T]; \mathcal{U}^1)$, $\omega \in C^2([0, T]; \mathcal{U}^0)$, $q \in C^1([0, T]; \mathcal{Y})$ from the relations

$$u''(t) = S_1 u'(t) + S_0 u(t) + (A^1)^{-1} Q \chi(t) q(t) + (A^1)^{-1} Q f(t), \quad (5)$$

$$u(0) = u_0, \quad u'(0) = u_1, \quad (6)$$

$$Cu(t) = \Psi(t) \equiv Cv(t), \quad (7)$$

$$H_0 \omega''(t) = H_1 \omega'(t) + \omega(t) + (B_0^0)^{-1} (I - Q) \chi(t) q(t) + (B_0^0)^{-1} (I - Q) f(t), \quad (8)$$

$$\omega(0) = \omega_0, \quad \omega'(0) = \omega_1, \quad (9)$$

where $S_1 = (A^1)^{-1} B_1^1$, $S_0 = (A^1)^{-1} B_0^1$, $u_0 = Pv_0$, $u_1 = Pv_1$, $\omega_0 = (I - P)v_0$, $\omega_1 = (I - P)v_1$, $t \in [0, T]$. The inverse problem (5) – (7) is called regular, and problem (8), (9) is called singular.

2. Solution of the Regular Inverse Problem

Rewrite problem (5) – (7) in the notation [18]. Let \mathcal{X}, \mathcal{Y} be Banach spaces, operators $S_1, S_0 \in Cl(\mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, operator-function $\Phi : [0, T] \rightarrow \mathcal{L}(\mathcal{Y}; \mathcal{X})$, functions $h : [0, T] \rightarrow \mathcal{X}$, $\Psi : [0, T] \rightarrow \mathcal{Y}$

$$u''(t) = S_1 u'(t) + S_0 u(t) + \Phi(t) q(t) + h(t), \quad t \in [0, T], \quad (10)$$

$$u(0) = u_0, \quad u'(0) = u_1, \quad (11)$$

$$Cu(t) = \Psi(t). \quad (12)$$

Theorem 2. Let the pencil \vec{B} be polynomially A -bounded and condition (A) be fulfilled, moreover, $C \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $\Phi \in C^1([0, T]; \mathcal{L}(\mathcal{Y}; \mathcal{X}))$, $h \in C^1([0, T]; \mathcal{X})$, $\Psi \in C^3([0, T]; \mathcal{Y})$, for any $t \in [0, T]$ the operator $C\Phi(t)$ be invertible and $(C\Phi)^{-1} \in C^1([0, T]; \mathcal{L}(\mathcal{Y}))$.

If the compatibility condition $Cu_1 = \Psi'(0)$ is satisfied, then the solution to the inverse problem (10) – (12) exists and is unique in the class of functions $q \in C^1([0, T]; \mathcal{Y})$, $u \in C^2([0, T]; \mathcal{X})$.

Proof. Reduce problem (10) – (12) to the problem for the first order equation

$$v'(t) = Av(t) + Q(t)q(t) + F(t), \quad t \in [0, T], \quad (13)$$

$$v(0) = v_0, \quad (14)$$

$$Bv(t) = Z(t), \quad (15)$$

where $v(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$, $A = \begin{pmatrix} 0 & I \\ S_0 & S_1 \end{pmatrix}$, $Q(t) = \begin{pmatrix} 0 \\ \Phi(t) \end{pmatrix}$, $F(t) = \begin{pmatrix} 0 \\ h(t) \end{pmatrix}$,
 $v(0) = \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$, $v_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & C \end{pmatrix}$, $Z(t) = \Psi'(t)$.

Put $R(t) = -(C\Phi(t))^{-1}$. Therefore, all the conditions of Theorem 6.2.3 from [18], are satisfied, and the function $q(t)$ satisfies the integral equation

$$q(t) = q_0(t) + R(t) \left(CS_0 \int_0^t V_{12}(t-s)\Phi(s)q(s)ds + CS_1 \int_0^t V_{22}(t-s)\Phi(s)q(s)ds \right), \quad (16)$$

where

$$q_0(t) = -R(t) \left(\Psi''(t) - CS_0V_{11}(t)u_0 - CS_1V_{21}(t)u_0 - CS_0V_{12}(t)u_1 - CS_1V_{22}(t)u_1 - \right. \\ \left. - CS_0 \int_0^t V_{12}(t-s)h(s)ds - CS_1 \int_0^t V_{22}(t-s)h(s)ds - Ch(t) \right).$$

Thus, the solution to the inverse problem (13) – (15) exists and is unique in the class of functions $q \in C^1([0, T]; \mathcal{Y})$, $v \in C^1([0, T]; \mathcal{X}^2)$. Thus, we obtain that the solution to the regular inverse problem (10) – (12) exists and is unique, with $q \in C^1([0, T]; \mathcal{Y})$, $u \in C^2([0, T]; \mathcal{X})$. □

In order to obtain a solution to a singular problem, we need a greater smoothness of the function q from the solution of a regular problem.

Lemma 2. Let $n \in \mathbb{N}$, $V \in C^{n-1}([0, T]; \mathcal{L}(\mathcal{X}))$, $g \in C^n([0, T]; \mathcal{X})$. Then

$$\left(\int_0^t V(t-s)g(s)ds \right)^{(n)} = \sum_{k=0}^{n-1} V^{(n-k-1)}(t)g^{(k)}(0) + \int_0^t V(t-s)g^{(n)}(s)ds. \quad (17)$$

Proof. Apply the method of mathematical induction. First we show that for $n = 1$ formula (17) is true. Find the first derivative with respect to t

$$\left(\int_0^t V(t-s)g(s)ds \right)'_t = V(0)g(t) + \int_0^t \frac{d}{dt} [V(t-s)] g(s)ds = V(0)g(t) - \\ - \int_0^t \frac{d}{ds} [V(t-s)] g(s)ds = V(t)g(0) + \int_0^t V(t-s)g'(s)ds. \quad (18)$$

Comparing (17) and (18) we conclude that formula (17) for $n = 1$ is true. Now show that if (17) is true for n , then it is also true for $n + 1$:

$$\begin{aligned} \left(\left(\int_0^t V(t-s)g(s)ds \right)^{(n)} \right)' &= \sum_{k=0}^{n-1} V^{(n-k)}(t)g^{(k)}(0) + V(t)g^{(n)}(0) + \int_0^t V(t-s)g^{(n+1)}(s)ds = \\ &= \sum_{k=0}^n V^{(n-k)}(t)g^{(k)}(0) + \int_0^t V(t-s)g^{(n+1)}(s)ds = \left(\int_0^t V(t-s)g(s)ds \right)^{(n+1)}. \end{aligned}$$

□

The following theorem provides sufficient conditions for the existence of a more smooth (as $p \in \mathbb{N}$) solution $q \in C^{p+2}([0, T], \mathcal{Y})$ of a regular problem.

Theorem 3. *Let the pencil \vec{B} be polynomially A -bounded and condition (A) be fulfilled, moreover, $C \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $p \in \mathbb{N}_0$, $\Phi \in C^{p+2}([0, T]; \mathcal{L}(\mathcal{Y}; \mathcal{X}))$, $h \in C^{p+2}([0, T]; \mathcal{X})$, $\Psi \in C^{p+4}([0, T]; \mathcal{Y})$, for any $t \in [0, T]$ operator $C\Phi(t)$ be invertible, with $(C\Phi)^{-1} \in C^{p+2}([0, T]; \mathcal{L}(\mathcal{Y}))$ and the compatibility condition $Cu_1 = \Psi'(0)$ be satisfied for some $u_1 \in \mathcal{U}$. Then there exists and a unique solution of (10) – (12), and $q \in C^{p+2}([0, T]; \mathcal{Y})$.*

Proof. Denote by $V_{11} = \frac{1}{2\pi i} \int_{\gamma} (\mu^2 A - \mu B_1 - B_0)^{-1} (\mu A - B_1) e^{\mu t} d\mu$,

$$V_{12} = \frac{1}{2\pi i} \int_{\gamma} (\mu^2 A - \mu B_1 - B_0)^{-1} A e^{\mu t} d\mu,$$

$$V_{21} = \frac{1}{2\pi i} \int_{\gamma} (\mu^2 A - \mu B_1 - B_0)^{-1} B_0 e^{\mu t} d\mu,$$

$$V_{22} = \frac{1}{2\pi i} \int_{\gamma} (\mu^2 A - \mu B_1 - B_0)^{-1} \mu A e^{\mu t} d\mu$$

the propagators of the homogeneous equation (10). Earlier, in the proof of Theorem 2, it was established that the function $q(t)$ satisfies the integral equation (16). Take the natural number $n \leq p + 2$. Assuming that $q \in C^n([0, T]; \mathcal{Y})$ by Lemma 2 we obtain the equality

$$\begin{aligned} q^{(n)}(t) &= q_0^{(n)}(t) + \sum_{k=0}^{n-1} C_n^k R^{(k)}(t) C S_0 \sum_{m=0}^{n-k-1} V_{12}^{(n-k-m-1)}(t) (\Phi q)^{(m)}(0) + \\ &+ \sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) q^{(n-k-m)}(s) ds + \\ &+ \sum_{k=0}^{n-1} C_n^k R^{(k)}(t) C S_1 \sum_{m=0}^{n-k-1} V_{22}^{(n-k-m-1)}(t) (\Phi q)^{(m)}(0) + \\ &+ \sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_1 \int_0^t V_{22}(t-s) \Phi^{(m)}(s) q^{(n-k-m)}(s) ds, \end{aligned}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$, $C_n^{k,m} = \frac{n!}{k!m!(n-k-m)!}$ and

$$\begin{aligned} q_0^{(n)}(t) &= - \sum_{k=0}^n C_n^k R^{(k)}(t) \left(\Psi^{(n-k+2)}(t) - C S_0 V_{11}^{(n-k)}(t) u_0 - \right. \\ &\left. - C S_1 V_{21}^{(n-k)}(t) u_0 - C S_0 V_{12}^{(n-k)}(t) u_1 - C S_1 V_{22}^{(n-k)}(t) u_1 - \right) \end{aligned}$$

$$\begin{aligned}
 & -CS_0 \int_0^t V_{12}(t-s)h^{(n-k)}(s)ds - CS_1 \int_0^t V_{22}(t-s)h^{(n-k)}(s)ds - \\
 & -Ch^{(n-k)}(t) \Big) + \sum_{k=0}^{n-1} C_n^k R^{(k)}(t)CS_0 \sum_{m=0}^{n-k-1} V_{12}^{(n-k-m-1)}(t)h^{(m)}(0) + \\
 & + \sum_{k=0}^{n-1} C_n^k R^{(k)}(t)CS_1 \sum_{m=0}^{n-k-1} V_{22}^{(n-k-m-1)}(t)h^{(m)}(0)
 \end{aligned}$$

exists from the conditions of this theorem for $n = 0, 1, \dots, p + 2$.

Show that $q \in C^{p+2}([0, T], \mathcal{Y})$, for this purpose denote $r_0 = q_0(0)$, and for $n = 1, 2, \dots, p + 2$ determine the following values

$$\begin{aligned}
 r_n = & q_0^{(n)}(0) + \sum_{k=0}^{n-1} C_n^k R^{(k)}(0)CS_0 \sum_{m=0}^{n-k-1} V_{12}^{(n-k-m-1)}(0) \sum_{l=0}^m C_m^l \Phi^{(l)}(0)r_{m-l} + \\
 & + \sum_{k=0}^{n-1} C_n^k R^{(k)}(0)CS_1 \sum_{m=0}^{n-k-1} V_{22}^{(n-k-m-1)}(0) \sum_{l=0}^m C_m^l \Phi^{(l)}(0)r_{m-l}.
 \end{aligned}$$

Consider the system of integral equations

$$\begin{aligned}
 \tilde{q}_0(t) = & q_0(t) + R(t) \left(CS_0 \int_0^t V_{12}(t-s)\Phi(s)\tilde{q}_0(s)ds + CS_1 \int_0^t V_{22}(t-s)\Phi(s)\tilde{q}_0(s)ds \right), \\
 \tilde{q}_n(t) = & q_0^{(n)}(t) + \sum_{k=0}^{n-1} C_n^k R^{(k)}(t)CS_0 \sum_{m=0}^{n-k-1} V_{12}^{(n-k-m-1)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0)r_{m-l} + \\
 & + \sum_{k=0}^{n-1} C_n^k R^{(k)}(t)CS_1 \sum_{m=0}^{n-k-1} V_{22}^{(n-k-m-1)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0)r_{m-l} + \\
 & + \sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t)CS_0 \int_0^t V_{12}(t-s)\Phi^{(m)}(s)\tilde{q}_{n-k-m}(s)ds + \\
 & + \sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t)CS_1 \int_0^t V_{22}(t-s)\Phi^{(m)}(s)\tilde{q}_{n-k-m}(s)ds, \quad n = 1, \dots, p + 2.
 \end{aligned} \tag{19}$$

Reduce (19) to the Volterra equation of the second kind

$$g(t) = g_0(t) + \int_0^t K(t, s)g(s)ds$$

on the space $(C([0, T]; \mathcal{Y}))^{p+3}$ with a matrix operator function $K(t, s)$ given on the triangle $\Delta = \{(t, s) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq s \leq t\}$. By virtue of the continuity of all data of system (19), it has a unique solution

$$(\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_{p+2}) \in (C([0, T]; \mathcal{Y}))^{p+3}.$$

This solution will be the limit of the sequence of approximations

$$\tilde{q}_{0,i}(t) = q_0(t) + R(t) \left(CS_0 \int_0^t V_{12}(t-s)\Phi(s)\tilde{q}_{0,i-1}(s)ds + CS_1 \int_0^t V_{22}(t-s)\Phi(s)\tilde{q}_{0,i-1}(s)ds \right),$$

$$\begin{aligned} \tilde{q}_{n,i}(t) &= q_0^{(n)}(t) + \sum_{k=0}^{n-1} C_n^k R^{(k)}(t) C S_0 \sum_{m=0}^{n-k-1} V_{12}^{(n-k-m-1)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0) r_{m-l} + \\ &+ \sum_{k=0}^{n-1} C_n^k R^{(k)}(t) C S_1 \sum_{m=0}^{n-k-1} V_{22}^{(n-k-m-1)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0) r_{m-l} + \\ &+ \sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m,i-1}(s) ds + \\ &+ \sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_1 \int_0^t V_{22}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m,i-1}(s) ds, \quad n = 1, \dots, p+2, \quad i \in \mathbb{N}, \end{aligned} \quad (20)$$

which for $i \rightarrow \infty$ on the interval $[0, T]$ converge uniformly to the functions \tilde{q}_n , $n = 0, \dots, p+2$. Set the initial approximation $\tilde{q}_{n,0} \equiv 0$, $n = 0, \dots, p+2$, then $\tilde{q}_{n+1,0} = \tilde{q}'_{n,0}$, $n = 0, \dots, p+1$. In addition, from (20) it follows that

$$\tilde{q}_{n,i}(0) = r_n, \quad n = 0, \dots, p+2, \quad i \in \mathbb{N}. \quad (21)$$

Assume that for all $j = 1, \dots, i$ the equalities $\tilde{q}_{n+1,j}(t) = \tilde{q}'_{n,j}(t)$, $n = 0, \dots, p+1$ are true. Then, using Lemma 2 and equalities (21), we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m,i}(s) ds \right) = \\ &= \sum_{k=1}^{n+1} \sum_{m=0}^{n-k+1} C_n^{k-1,m} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m+1,i}(s) ds + \\ &+ \sum_{k=0}^n C_n^k R^{(k)}(t) C S_0 V_{12}(t) \sum_{m=0}^{n-k} C_{n-k}^m \Phi^{(m)}(0) r_{n-k-m} + \\ &+ \sum_{k=0}^n \sum_{m=1}^{n-k+1} C_n^{k,m-1} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m+1,i}(s) ds + \\ &+ \sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m+1,i}(s) ds. \end{aligned} \quad (22)$$

Denote by $a_{k,m}$ some values in (22). Taking into account the equalities

$$C_n^k + C_n^{k-1} = C_{n+1}^k, \quad C_n^{k,m} + C_n^{k-1,m} + C_n^{k,m-1} = C_{n+1}^{k,m}$$

we get

$$\begin{aligned} &\sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} a_{k,m} + \sum_{k=1}^{n+1} \sum_{m=0}^{n-k+1} C_n^{k-1,m} a_{k,m} + \sum_{k=0}^n \sum_{m=1}^{n-k+1} C_n^{k,m-1} a_{k,m} = \\ &= \sum_{k=1}^n \sum_{m=1}^{n-k} C_{n+1}^{k,m} a_{k,m} + \sum_{k=1}^n C_{n+1}^{k,0} a_{k,0} + \sum_{m=1}^n C_{n+1}^{0,m} a_{0,m} + \sum_{k=1}^n C_{n+1}^{k,0} a_{k,n-k+1} + \\ &+ C_{n+1}^{0,0} a_{0,0} + C_{n+1}^{0,n} a_{0,n+1} + C_{n+1}^{n,0} a_{n+1,0} = \sum_{k=0}^{n+1} \sum_{m=0}^{n-k+1} C_{n+1}^{k,m} a_{k,m}. \end{aligned} \quad (23)$$

From (22) and (23) it follows that

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m,i}(s) ds \right) = \\ & = \sum_{k=0}^{n+1} \sum_{m=0}^{n-k+1} C_{n+1}^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{12}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m+1,i}(s) ds + \\ & \quad + \sum_{k=0}^n C_n^k R^{(k)}(t) C S_0 V_{12}(t) \sum_{m=0}^{n-k} C_{n-k}^m \Phi^{(m)}(0) r_{n-k-m}. \end{aligned} \quad (24)$$

Similarly, we can get that

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{k=0}^n \sum_{m=0}^{n-k} C_n^{k,m} R^{(k)}(t) C S_1 \int_0^t V_{22}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m,i}(s) ds \right) = \\ & = \sum_{k=0}^{n+1} \sum_{m=0}^{n-k+1} C_{n+1}^{k,m} R^{(k)}(t) C S_1 \int_0^t V_{22}(t-s) \Phi^{(m)}(s) \tilde{q}_{n-k-m+1,i}(s) ds + \\ & \quad + \sum_{k=0}^n C_n^k R^{(k)}(t) C S_1 V_{22}(t) \sum_{m=0}^{n-k} C_{n-k}^m \Phi^{(m)}(0) r_{n-k-m}. \end{aligned} \quad (25)$$

Changing the summation indices and re-grading the sums we get

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{k=0}^{n-1} C_n^k R^{(k)}(t) C S_0 \sum_{m=0}^{n-k-1} V_{12}^{(n-k-m-1)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0) r_{m-l} \right) = \\ & = \sum_{k=0}^n C_{n+1}^k R^{(k)}(t) C S_0 \sum_{m=0}^{n-k} V_{12}^{(n-k-m)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0) r_{m-l} - \\ & \quad - \sum_{k=0}^n C_n^k R^{(k)}(t) C S_0 V_{12}(t) \sum_{m=0}^{n-k} C_{n-k}^m \Phi^{(m)}(0) r_{n-k-m}. \end{aligned} \quad (26)$$

Similarly, we get that

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{k=0}^{n-1} C_n^k R^{(k)}(t) C S_1 \sum_{m=0}^{n-k-1} V_{22}^{(n-k-m-1)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0) r_{m-l} \right) = \\ & = \sum_{k=0}^n C_{n+1}^k R^{(k)}(t) C S_1 \sum_{m=0}^{n-k} V_{22}^{(n-k-m)}(t) \sum_{l=0}^m C_m^l \Phi^{(l)}(0) r_{m-l} - \\ & \quad - \sum_{k=0}^n C_n^k R^{(k)}(t) C S_1 V_{22}(t) \sum_{m=0}^{n-k} C_{n-k}^m \Phi^{(m)}(0) r_{n-k-m}. \end{aligned} \quad (27)$$

Differentiating (20), and also using (24) – (27), we obtain the equalities $\tilde{q}'_{n,i+1} = \tilde{q}'_{n+1,i+1}$, $n = 0, \dots, p+1$. Thus, the sequence $\tilde{q}_{0,i}$ converges as $i \rightarrow \infty$ to the function \tilde{q}_0 uniformly on the interval $[0, T]$, and the sequence $\tilde{q}'_{0,i} = \tilde{q}'_{1,i}$ converges as $i \rightarrow \infty$ to the function \tilde{q}'_1 uniformly on the segment $[0, T]$. Therefore, the function \tilde{q}_0 is continuously differentiable and $\tilde{q}'_0 = \tilde{q}'_1$. The equalities of $\tilde{q}'_n = \tilde{q}'_{n+1}$, $n = 1, \dots, p+1$, are proved in the same way, which implies that $\tilde{q}_0 \equiv q \in C^{p+2}([0, T]; \mathcal{Y})$ and, therefore, $q^{(n)} = \tilde{q}_n$, $n = 1, \dots, p+2$. □

3. Solvability of the Original Inverse Problem

Now we can formulate the solvability conditions for the original problem (1) – (4).

Theorem 4. *Let the pencil \vec{B} be polynomially A -bounded and condition (A) be fulfilled, moreover, the ∞ be a pole of order $p \in \mathbb{N}_0$ of the A -resolvent of the pencil \vec{B} , operator $C \in \mathcal{L}(\mathcal{U}; \mathcal{Y})$, $\mathcal{U}^0 \subset \ker C$, $\chi \in C^{p+2}([0, T]; \mathcal{L}(\mathcal{Y}; \mathcal{F}))$, $f \in C^{p+2}([0, T]; \mathcal{F})$, $\Psi \in C^{p+4}([0, T]; \mathcal{Y})$, for any $t \in [0, T]$ operator $C(A^1)^{-1}Q\chi$ be invertible, with $(C(A^1)^{-1}Q\chi)^{-1} \in C^{p+2}([0, T]; \mathcal{L}(\mathcal{Y}))$, the condition $Cu_1 = \Psi'(0)$ be satisfied at some initial value $u_1 = v_1 \in \mathcal{U}$, and the initial values $w_k = (I - P)v_k \in \mathcal{U}^0$ satisfy*

$$w_k = - \sum_{j=0}^p K_j^2 (B_0^0)^{-1} \frac{d^{j+k}}{dt^{j+k}} \left[(I - Q)(\chi(0)q(0) + f(0)) \right], \quad k = 0, 1.$$

Then there exists a unique solution (v, q) of inverse problem (1) – (4), where $q \in C^{p+2}([0, T]; \mathcal{Y})$, $v = u + w$, whence $u \in C^2([0, T]; \mathcal{U}^1)$ is the solution of (5) – (7) and the function $w \in C^2([0, T]; \mathcal{U}^0)$ is a solution of (8), (9) given by

$$w(t) = - \sum_{j=0}^p K_j^2 (B_0^0)^{-1} \frac{d^j}{dt^j} \left[(I - Q)(\chi(t)q(t) + f(t)) \right]. \quad (28)$$

Proof. The conditions of Theorems 2 and 3 are satisfied, and therefore there exists a unique solution (q, u) of problem (5) – (7), where $q \in C^{p+2}([0, T]; \mathcal{Y})$, $u \in C^2([0, T]; \mathcal{U}^1)$.

Using the result of [17] and the required smoothness of the function q , we obtain that there exists a unique solution $w \in C^2([0, T]; \mathcal{U}^0)$ to (8), (9) given by (28).

□

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ОБРАТНАЯ ЗАДАЧА ДЛЯ МАТЕМАТИЧЕСКИХ МОДЕЛЕЙ СОБОЛЕВСКОГО ТИПА*А.А. Замышляева¹, А.В. Лут¹*¹ Южно-Уральский государственный университет, г. Челябинск, Российская Федерация

Работа посвящена исследованию обратной задачи для линейного уравнения соболевского типа высокого порядка с неизвестным коэффициентом, зависящим от времени. Поскольку уравнение может быть вырожденным, используется метод фазового пространства. Он состоит в построении проекторов, расщепляющих исходные пространства в прямую сумму подпространств. Действия операторов также расщепляются. Таким образом, исходная модель сводится к двум задачам: регулярной и сингулярной. Регулярная редуцируется к невырожденной задаче первого порядка, которая решается с помощью аппроксимаций. Получена необходимая гладкость решения. Затем оно подставляется в сингулярную задачу, которая решается с использованием методов теории относительно полиномиально ограниченных пучков операторов. Основным результатом работы содержатся достаточные условия существования и единственности решения обратной задачи для математической модели соболевского типа второго порядка. Данная методика может быть использована при исследовании обратных задач рассматриваемого типа для математической модели Буссинеска – Лява.

Ключевые слова: уравнения соболевского типа; обратная задача; математические модели; уравнения второго порядка.

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