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EXPONENTIAL DICHOTOMIES IN BARENBLATT–ZHELTOV–KOCHINA MODEL IN SPACES OF DIFFERENTIAL FORMS WITH "NOISE"

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We investigate stability of solutions in linear stochastic Sobolev type models with the relatively bounded operator in spaces of smooth differential forms defined on smooth compact oriented Riemannian manifolds without boundary. To this end, in the space of differential forms, we use the pseudo-differential Laplace–Beltrami operator instead of the usual Laplace operator. The Cauchy condition and the Showalter–Sidorov condition are used as the initial conditions. Since "white noise" of the model is non-differentiable in the usual sense, we use the derivative of stochastic process in the sense of Nelson–Gliklikh. In order to investigate stability of solutions, we establish existence of exponential dichotomies dividing the space of solutions into stable and unstable invariant subspaces. As an example, we use a stochastic version of the Barenblatt–Zheltov–Kochina equation in the space of differential forms defined on a smooth compact oriented Riemannian manifold without boundary.

Keywords: Sobolev type equations; differential forms; stochastic equations; Nelson–Gliklikh derivative.

Dedicated to the 60-th birthday
of outstanding mathematician Jacek Banasiak.

Introduction

The Barenblatt–Zheltov–Kochina equation

$$(\lambda - \Delta)\dot{u} = \alpha\Delta u + f \tag{1}$$

simulates dynamics of pressure of a liquid filtered in a fractured–porous medium. Parameters α and λ are real and characterize the environment and properties of the liquid, respectively, and function $f = f(x)$ plays the role of an external influence.

The study of solvability of the initial–boundary value problems for equation (1) in Banach spaces with the Cauchy condition is based on the approach described, for example, in [1], where this equation is reduced to abstract linear Sobolev type equation

$$L\dot{u} = Mu + f \tag{2}$$

in suitable function spaces $\mathfrak{U}$ and $\mathfrak{F}$. Paper [2] considers splitting of similar spaces and splitting of the action of elliptic operators in spaces of smooth differential forms defined on smooth Riemannian manifolds without boundary.

Further, paper [3] considers stochastic equations of Sobolev type

$$L \overset{o}{\eta} = M\eta + N\omega, \tag{3}$$

where $\eta = \eta(t)$ is a stochastic process, $\overset{o}{\eta}$ is the Nelson–Gliklikh derivative of process [4], $w = w(t)$ is a stochastic process that corresponds to an external influence; $L, M, N \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ are operators, and operator M is (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Paper [5] shows the results of studying equation (3) in the case when operator M is (L, p)-sectorial, $p \in \{0\} \bigcup \mathbb{N}$. Then, paper [6] considers equation (3) in the case when operator M is (L, p)-radial, $p \in \{0\} \bigcup \mathbb{N}$. Note that all three papers [3,5,6] along with classical Cauchy problem

$$\eta(0) = \eta_0 \tag{4}$$

consider Showalter–Sidorov problem

$$P(\eta(0) - \eta_0) = 0. \tag{5}$$

Paper [7] considers more general initial-finite conditions for equation (3), and paper [8] investigates the Cauchy and Showalter–Sidorov problems posed for the Sobolev type equation of high order.

The stochastic Barenblatt–Zheltov–Kochina model with additive "white noise" given in a bounded domain is considered as a concrete interpretation of abstract stochastic equation (3) in [3] and is transferred to a Riemannian manifold without boundary in [9]. Paper [10] was the first to consider the dichotomies of solutions to abstract homogeneous Sobolev type equation

$$L\dot{u} = Mu, \tag{6}$$

where operator M is (L, p)-bounded, $p \in \{0\} \bigcup \mathbb{N}$.

The paper is devoted to the study of dichotomy of solutions to abstract homogeneous stochastic Sobolev type equation

$$L \overset{o}{\eta} = M\eta, \tag{7}$$

where operator M is (L, p)-bounded, $p \in \{0\} \bigcup \mathbb{N}$. The Barenblatt–Zheltov–Kochina stochastic equation given on a Riemannian manifold without boundary is considered as a specific model.

Note also other approaches to the study of groups and semigroups of stochastic equations, for example, proposed in [11] or in [12–14].

The paper, in addition to the introduction, conclusion and bibliography, contains three sections. In the first section, we introduce the terminology of linear Sobolev type equations, including the terminology related to stability, and the main theorems proved in other papers. The second section describes a stochastic analogue of the Barenblatt–Zheltov–Kochina equation in specially selected spaces. The third section gives the main result on existence of exponential dichotomies of solutions to the equations. The conclusion presents several directions for further research. Note also that the bibliography does not pretend to be complete, but reflects only the personal preferences of the authors.

1. Dichotomies of Solutions to Equations in Banach Spaces

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, and operators $L,\ M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. Consider L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of operator M. If L-spectrum $\sigma^L(M)$ of operator M is bounded, then operator M is called (L, σ)-bounded. If operator M is (L, σ)-bounded, then there exist projectors

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} L^L_\mu(M)d\mu \in \mathcal{L}(\mathfrak{F}).$$

Here $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are *the right* and *the left L-resolvents* of operator M, respectively, and closed loop $\gamma \subset \mathbb{C}$ bounds the domain containing $\sigma^L(M)$. Set $\mathfrak{U}^0$ $(\mathfrak{U}^1) = \ker P$ $(\mathrm{im} P)$, $\mathfrak{F}^0$ $(\mathfrak{F}^1) = \ker Q$ $(\mathrm{im} Q)$. Denote the restriction of operator L (M) to $\mathfrak{U}^k$ by L_k (M_k), $k = 0, 1$.

Theorem 1. [1, Ch.3] *Let M be (L,σ)-bounded operator, then*
(i) operators L_k $(M_k) \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;
(ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$.

Corollary 1. *Let M be (L,σ)-bounded operator, then*

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty} \mu^k S^{k-1} L_1^{-1} Q + \sum_{k=1}^{\infty} \mu^{-k} H^k M_0^{-1}(\mathbb{I} - Q),$$

where operator $H = L_0^{-1}M_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Further, (L,σ)-bounded operator M is called (L,p)-*bounded*, $p \in \{0\} \cup \mathbb{N}$, if ∞ is a removable singular point $(H \equiv \mathbb{O},\ p = 0)$ or a pole of order $p \in \mathbb{N}$ (i.e. $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of *L-resolvents* $(\mu L - M)^{-1}$ of operator M. We consider vector function $u \in C^1(\mathbb{R}; \mathfrak{U})$ as *a solution* of equation (6), if u substituted to equation (6) makes (6) true. Solution $u = u(t)$ of equation (6) is called *a solution of Cauchy problem*

$$u(0) = u_0 \tag{8}$$

for equation (6), if equality (8) holds for some $u_0 \in \mathfrak{U}$.

Definition 1. *Set $\mathfrak{P} \subset \mathfrak{U}$ is called a phase space of equation (6), if*
(i) any solution $u = u(t)$ to equation (6) belongs to $\mathfrak{P}$ pointwise, i.e. $u(t) \in \mathfrak{P}$ for all $t \in \mathbb{R}$;
(ii) for any $u_0 \in \mathfrak{P}$ there exists unique solution $u \in C^1(\mathbb{R}; \mathfrak{U})$ of Cauchy problem (8) of equation (6).

Theorem 2. [1, Ch.3] *Let M be (L,p)-bounded operator, $p \in \{0\} \cup \mathbb{N}$. Then phase space of equation (6) is the subspace $\mathfrak{U}^1$.*

Note that if there exists operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$, then phase space of equation (6) is space $\mathfrak{U}$.

Definition 2. [10] *Subspace $\mathfrak{I} \subset \mathfrak{U}$ is called an invariant space of equation (6), if for any $u_0 \in \mathfrak{I}$ the solution of problem (6), (8) is $u \in C^1(\mathbb{R}; \mathfrak{I})$.*

Note that if equation (6) has phase space $\mathfrak{P}$, then any its invariant space $\mathfrak{I} \subset \mathfrak{P}$.

Definition 3. [10] *Solution $u = u(t)$ of equation (6) has exponential dichotomy , if*
(i) phase space $\mathfrak{P}$ of equation (6) splits into a direct sum of two invariant spaces (i.e. $\mathfrak{P} = \mathfrak{I}^+ \oplus \mathfrak{I}^-$), and
(ii) there exist constants $N_k \in \mathbb{R}_+, \nu_k \in \mathbb{R}_+, k = 1, 2,$ such that

$$\|u^1(t)\|_{\mathfrak{U}} \le N_1 e^{-\nu_1(s-t)}\|u^1(s)\|_{\mathfrak{U}} \quad for \quad s \ge t,$$
$$\|u^2(t)\|_{\mathfrak{U}} \le N_2 e^{-\nu_2(s-t)}\|u^2(s)\|_{\mathfrak{U}} \quad for \quad t \ge s,$$

where $u^1 = u^1(t) \in \mathfrak{I}^+$ *and* $u^2 = u^2(t) \in \mathfrak{I}^-$ *for any* $t \in \mathbb{R}$. *Space* $\mathfrak{I}^+(\mathfrak{I}^-)$ *is called stable (unstable) invariant space of equation (6). And if* $\mathfrak{I}^+ = \mathfrak{P}$ $(\mathfrak{I}^- = \mathfrak{P})$, *then stationary solution of equation (6) is stable (unstable).*

Theorem 3. [10] *Let* M *be* (L,p)*-bounded operator,* $p \in \{0\} \cup \mathbb{N}$, *and* $\sigma^L(M) \bigcap \{i\mathbb{R}\} = \emptyset$. *Then solution* $u = u(t)$ *of equation (6) has exponential dichotomy.*

2. Dichotomies of Solutions of Equations in Spaces of Differentiable "Noises"

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a full probability space, $\mathbb{R}$ be set of real numbers endowed with the Borel σ-algebra. Measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *a random variable.* A set of random variables having zero expectation ($\mathbf{E}\xi = 0$) and finite dispersion forms Hilbert space $\mathbf{L_2}$ with scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. Let $\mathcal{A}_0$ be a σ-subalgebra of σ-algebra $\mathcal{A}$. Construct subspace $\mathbf{L_2^0} \subset \mathbf{L_2}$ of random variables measurable with respect to $\mathcal{A}_0$. Denote the orthoprojector by $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$. Let $\xi \in \mathbf{L_2}$, then $\Pi\xi$ is called *a conditional expectation* of the random variable ξ, and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$.

Consider set $\mathfrak{I} \subset \mathbb{R}$ and the following two mappings. First, $f : \mathfrak{I} \to \mathbf{L_2}$, associates each $t \in \mathfrak{I}$ with random variable $\xi \in \mathbf{L_2}$. Second, $g : \mathbf{L_2} \times \Omega \to \mathbb{R}$, associates each pair (ξ, ω) with point $\xi(\omega) \in \mathbb{R}$. The mapping $\eta : \mathbb{R} \times \Omega \to \mathbb{R}$ having form $\eta = \eta(t, \omega) = g(f(t), \omega)$ is called *(one–dimensional) stochastic process.* Therefore, stochastic process $\eta = \eta(t, \cdot)$ is a random variable for each fixed $t \in \mathfrak{I}$, i.e. $\eta(t, \cdot) \in \mathbf{L_2}$, and stochastic process $\eta = \eta(\cdot, \omega)$ is called *a (sample) path* for each fixed $\omega \in \Omega$. Stochastic process η is called *continuous,* if all its paths are almost sure continuous (i.e. at almost all $\omega \in \Omega$ paths $\eta(\cdot, \omega)$ are continuous). The set of continuous stochastic processes form a Banach space, which we denote by $\mathbf{CL_2}$. Fix $\eta \in \mathbf{CL_2}$ and $t \in \mathfrak{I}$, and denote by $\mathcal{N}_t^\eta$ the σ-algebra generated by a random variable $\eta(t)$. For brevity, $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$.

Definition 4. *Let* $\eta \in \mathbf{CL_2}$. *A random variable*

$$\overset{o}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \boldsymbol{E}_t^\eta\left(\frac{\eta(t+\Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \boldsymbol{E}_t^\eta\left(\frac{\eta(t, \cdot) - \eta(t-\Delta t, \cdot)}{\Delta t}\right)\right)$$

is called Nelson–Gliklikh derivative $\overset{o}{\eta}$ *of the stochastic process* η *at point* $t \in \mathfrak{I}$, *if the limits exist in the sense of the uniform metric on* $\mathbb{R}$.

Let $\mathbf{C}^l\mathbf{L}_2$, $l \in \mathbb{N}$, be a space of stochastic processes whose paths are almost sure differentiable in the sense of the Nelson–Gliklikh derivative on $\mathfrak{I}$ up to order l inclusively. Spaces $\mathbf{C}^l\mathbf{L}_2$ are called *the spaces of differentiable "noises".* Let $\mathfrak{I} = \{0\} \cup \mathbb{R}_+$, then a well–known example [3, 5] of a vector of space $\mathbf{C}^l\mathbf{L}_2$ is given by a stochastic process that describes the Brownian motion in Einstein–Smoluchowski model

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin\frac{\pi}{2}(2k+1)t,$$

where independent random variables $\xi_k \in \mathbf{L}_2$ are such that dispersion $D\xi_k = [\frac{\pi}{2}(2k+1)]^{-2}$, $k \in \{0\} \cup \mathbb{N}$. As shown in [4], $\overset{o}{\beta}(t) = \frac{\beta(t)}{2t}$, $t \in \mathbb{R}_+$.

Now let $\mathfrak{U}$ ($\mathfrak{F}$) be a real separable Hilbert space with orthonormal basis $\{\varphi_k\}(\{\psi_k\})$. Denote by $\mathbf{U_L L}_2$ ($\mathbf{F_M L}_2$) the Hilbert space, which is a completion of the linear span of *random* $\mathbf{L}$*-variables*

$$\eta = \sum_{k=1}^{\infty} \lambda_k u_k \xi_k \varphi_k \quad (\omega = \sum_{k=1}^{\infty} \mu_k f_k \zeta_k \psi_k) \tag{9}$$

by the norm

$$\|\eta\|^2_{\mathbf{U}} = \sum_{k=1}^{\infty} \lambda_k^2 u_k^2 \mathbf{D}\xi_k \quad (\|\omega\|^2_{\mathbf{F}} = \sum_{k=1}^{\infty} \mu_k^2 f_k^2 \mathbf{D}\zeta_k).$$

Here sequence $\mathbf{L} = \{\lambda_k\} \subset \mathbb{R}_+$ ($\mathbf{M} = \{\mu_\mathbf{k}\} \subset \mathbb{R}_+$) is such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$ ($\sum\limits_{k=1}^{\infty} \mu_k^2 + \infty$), $\{u_k\}$ ($\{f_k\}$) is a sequence of coefficients of vector $u \in \mathfrak{U}$ ($f \in \mathfrak{F}$) expansion by basis $\{\varphi_k\}$ ($\{\psi_k\}$), and $\{\xi_k\} \subset \mathbf{L}_2$ ($\{\zeta_k\} \subset \mathbf{L}_2$) is a sequence of random variables. Note that for existence of a random $\mathbf{L}$-variable $\eta \in \mathbf{U_L L}_2$ ($\omega \in \mathbf{F_M L}_2$) it is enough to consider a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$ ($\{\zeta_k\} \subset \mathbf{L}_2$) having uniformly bounded dispersions, i.e. $\mathbf{D}\xi_k \le const,\ k \in \mathbb{N}$ ($\mathbf{D}\zeta_k \le const,\ k \in \mathbb{N}$).

Next, consider interval $\mathfrak{I} = (\varepsilon, \tau) \subset \mathbb{R}$. Mapping $\eta : (\varepsilon, \tau) \to \mathbf{U_L L}_2$ given by formula

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k u_k \xi_k(t) \varphi_k, \tag{10}$$

where $\{\xi_k\} \subset \mathbf{CL}_2$ is a sequence, is called $\mathfrak{U}$*-valued continuous stochastic* $\mathbf{L}$*-process*, if the series on the right-hand side converges uniformly on any compact in $\mathfrak{I}$ by norm $\|\cdot\|_\mathbf{U}$, and paths of process $\eta = \eta(t)$ are almost sure continuous. Continuous stochastic $\mathbf{L}$-process $\eta = \eta(t)$ is called *continuously Nelson–Gliklikh differentiable on* $\mathfrak{I}$, if series

$$\overset{o}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k u_k \overset{o}{\xi}_k(t) \varphi_k \tag{11}$$

converges uniformly on any compact in $\mathfrak{I}$ by norm $\|\cdot\|_\mathbf{U}$, and paths of process $\overset{o}{\eta} = \overset{o}{\eta}(t)$ are almost sure continuous. Let $\mathbf{C}(\mathfrak{I}, \mathbf{U_L L}_2)$ be a space of continuous stochastic $\mathbf{L}$-processes, and $\mathbf{C}^l(\mathfrak{I}, \mathbf{U_L L}_2)$ be a space of continuously differentiable up to order $l \in \mathbb{N}$ stochastic $\mathbf{L}$-processes. An example of a stochastic $\mathbf{L}$-process, which is continuously differentiable up to any order $l \in \mathbb{N}$ inclusively, is Wiener $\mathbf{L}$-process [3, 5]

$$W_\mathbf{L}(t) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t) \varphi_k,$$

where $\{\beta_k\} \subset \mathbf{C}^l\mathbf{L}_2$ is a sequence of Brownian motions on $\mathbb{R}_+$. Similarly, spaces $\mathbf{C}(\mathfrak{I}, \mathbf{F_M L}_2)$ and $\mathbf{C}^l(\mathfrak{I}, \mathbf{F_M L}_2)$, $l \in \mathbb{N}$, are constructed. Note also that spaces $\mathbf{C}_l\mathbf{L}_2, \mathbf{C}(\mathfrak{I}, \mathbf{U_L L}_2)$ and $\mathbf{C}^l(\mathfrak{I}, \mathbf{F_M L}_2)$, $l \in \mathbb{N}$, are called *the spaces of differentiable* $\mathbf{L}$*-"noises"* [3].

Consider an operator $A \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. It is clear that the same operator $A \in \mathcal{L}(\mathbf{U_L L}_2; \mathbf{F_M L}_2)$. Moreover, there exists the following lemma holds.

Lemma 1. *Let operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, where operator M is (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then operator $M \in \mathcal{L}(\mathbf{U_L L}_2; \mathbf{F_M L}_2)$ is also (L,p)-bounded,*

$p \in \{0\} \cup \mathbb{N}$, where operator $L \in \mathcal{L}(\mathbf{U_L L_2}; \mathbf{F_M L_2})$. Moreover, L-spectrum of operator $M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ coincides with L-spectrum of operator $M \in \mathcal{L}(\mathbf{U_L L_2}; \mathbf{F_M L_2})$.

The interested reader is encouraged to prove this statement. According to Lemma 2.1, all results of section 1 are transferred from Banach spaces to spaces of differentiable **L**-"noises".

Let operators $L,\ M \in \mathcal{L}(\mathbf{U_L L_2}; \mathbf{F_M L_2})$. Consider equation

$$L \overset{o}{\eta} = M\eta. \tag{12}$$

Stochastic **L**-process $\eta \in \mathbf{C}^1(\mathbb{R}; \mathbf{U_L L_2})$ is called *a solution of equation (12)*, if η substituted to equation (12) makes (12) true. Solution $\eta = \eta(t)$ of equation (12) is called *a solution of Cauchy problem*

$$\eta(0) = \eta_0, \tag{13}$$

for equation (12), if equality (13) holds for some random **L**-variable $\eta_0 \in \mathbf{U_L L_2}$.

Definition 5. *Set $\mathbf{P_L L_2} \subset \mathbf{U_L L_2}$ is called a stochastic phase space of equation (12), if*

(i) almost surely each path of solution $\eta = \eta(t)$ of equation (12) belong to $\mathbf{P_L L_2}$, i.e. $\eta(t) \in \mathbf{P_L L_2}, t \in \mathbb{R}$ for almost all paths;

(ii) there exists the unique solution of problem (12), (13) for almost all paths $\eta_0 \in \mathbf{P_L L_2}$.

Since the solution of problem (12), (13) is a stochastic **L**-process, and only one of its paths is observed in reality, we consider necessary to make an explanation. Recall [5–7] that stochastic **L**-processes $\eta = \eta(t)$ and $\zeta = \zeta(t)$ are considered equal, if almost surely each path of one of them coincides with any path of the other. Next, extend projector P of section 1 from Banach space $\mathfrak{U}$ to the space of random **L**-variables $\mathbf{U_L L_2}$. It is easy to show that operator $P \in \mathcal{L}(\mathbf{U_L L_2})$ is also a projector. Set $\mathbf{U^0_L L_2} = \ker P$, $\mathbf{U^1_L L_2} = \mathrm{im} P$ such that $\mathbf{U_L L_2} = \mathbf{U^0_L L_2} \oplus \mathbf{U^1_L L_2}$.

Theorem 4. *Let operators $L, M \in \mathcal{L}(\mathbf{U_L L_2}; \mathbf{F_M L_2})$, where operator M is (L, p)-bounded. Then phase space of equation (12) is space $\mathbf{U^1_L L_2}$.*

Note that if there exists operator $L^{-1} \in \mathcal{L}(\mathbf{F_M L_2}; \mathbf{U_L L_2})$, then $\mathbf{U^1_L L_2} = \mathbf{U_L L_2}$.

Definition 6. *Subspace $\mathbf{I_L L_2} \subset \mathbf{U_L L_2}$ is called an invariant space of equation (12), if for any $\eta_0 \in \mathbf{I_L L_2}$ the solution of problem (12), (13) is $\eta \in \mathbf{C}^1(\mathbb{R}; \mathbf{I_L L_2})$.*

Note that if equation (12) has phase space $\mathbf{P_L L_2}$ and invariant space $\mathbf{I_L L_2}$, then $\mathbf{I_L L_2} \subset \mathbf{P_L L_2}$.

Definition 7. *Solution $\eta = \eta(t)$ of equation (12) has exponential dichotomy, if*

(i) phase space $\mathbf{P_L L_2}$ of equation (12) splits into a direct sum of two invariant spaces (i.e. $\mathbf{P_L L_2} = \mathbf{I^+_L L_2} \oplus \mathbf{I^-_L L_2}$), and

(ii) there exist constants $N_k \in \mathbb{R}_+, \nu_k \in \mathbb{R}_+, k = 1, 2$, such that

$$\begin{aligned} \|\eta^1(t)\|_{\mathbf{U}} &\le N_1 e^{-\nu_1(s-t)} \|\eta^1(s)\|_{\mathbf{U}} \qquad \text{for} \qquad s \ge t, \\ \|\eta^2(t)\|_{\mathbf{U}} &\le N_2 e^{-\nu_2(s-t)} \|\eta^2(s)\|_{\mathbf{U}} \qquad \text{for} \qquad t \ge s, \end{aligned}$$

where $\eta^1 = \eta^1(t) \in \mathbf{I^+_L L_2}$ and $\eta^2 = \eta^2(t) \in \mathbf{I^-_L L_2}$ for all $t \in \mathbb{R}$. Space $\mathbf{I^+_L L_2}$ ($\mathbf{I^-_L L_2}$) is called the stable (unstable) invariant space of equation (12).

Theorem 5. *Let M be (L,p)-bounded operator, $p \in \{0\} \cup \mathbb{N}$, and $\sigma^L(M) \bigcap \{i\mathbb{R}\} = \emptyset$. Then solution $\eta = \eta(t)$ of equation (12) has exponential dichotomy.*

Let us give an idea of the proof. In order to define projectors in the space of random $\mathbf{L}$-variables $\mathbf{U_L L}_2$, we use formulas

$$P_1 = \frac{1}{2\pi i} \int\limits_{\Gamma_1} R^L_\mu(M) d\mu \in \mathcal{L}(\mathbf{U_L L}_2), \quad P_2 = \frac{1}{2\pi i} \int\limits_{\Gamma_2} R^L_\mu(M) d\mu \in \mathcal{L}(\mathbf{U_L L}_2),$$

where contour Γ_1 (Γ_2) belongs to the left (right) half-plane of the complex plane and bounds a part of L-spectrum of operator M

$$\sigma^L(M) \bigcap \{\mu : Re\mu < 0\} \quad (\sigma^L(M) \bigcap \{\mu : Re\mu > 0\}).$$

Set $\mathbf{I^+_L L}_2 = \mathrm{im} P_1$ and $\mathbf{I^-_L L}_2 = \mathrm{im} P_2$. Obviously, $\mathbf{U^1_L L}_2 = \mathbf{I^+_L L}_2 \oplus \mathbf{I^-_L L}_2$. Let $\eta^1 \in \mathbf{I^+_L L}_2$. If $s \geq t$, then

$$\|\eta^1(t)\|_{\mathbf{U}} \leq e^{-\nu_1(s-t)} \frac{1}{2\pi} \int\limits_{\Gamma'_1} |R^L_\tau(M)| \, |e^{\tau(s-t)}| \, |d\tau| \, \|\eta^1(s)\|_{\mathbf{U}} \leq e^{-\nu_1(s-t)} N_1 \|\eta^1(s)\|_{\mathbf{U}},$$

where $\tau = \mu + \nu_1$, and $Re\tau > 0$, $\tau \in \Gamma'_1$. The estimate for $\eta^2 \in \mathbf{I^-_L L}_2$ is obtained similarly.

3. Exponential Dichotomies of the Barenblatt–Zheltov–Kochina Stochastic Equation in Spaces of Differential Forms

Let Ω_n be a n-dimensional smooth compact oriented connected Riemannian manifold without boundary, and $E^q = E^q(\Omega_n), 0 \leq q \leq n$ be a space of differential q-forms on Ω_n. In particular, $E^0(\mathbb{R}^n)$ is a space of functions of n variables. Consider Laplace–Beltrami operator $\Delta : E^q \to E^q$, defined by equality $\Delta = \delta d + d\delta$, where $d : E^q \to E^{q+1}$ is the operator of external differential from differential forms, and $\delta : E^q \to E^{q-1}$ can be presented as linear Hodge operator $\delta = (-1)^{n(q+1)+1} * d*$, $* : E^q \to E^{n-q}$, which associates a q-form on Ω_n with $(n-q)$-form. Denote the space of harmonic q-forms by $H^q = \{\omega \in E^q : \Delta\omega = 0\}$.

It follows from the Hodge decomposition (see, for example, in [9])

$$E^q = \Delta(E^q) \oplus H^q = d\delta(E^q) \oplus \delta d(E^q) \oplus H^q \tag{14}$$

that equation $\Delta\omega = \alpha$ has solution $\omega \in E^q$, if a q-form α is orthogonal to space of harmonic forms H^q.

Define scalar product in space E^q, $q = 0, 1, \ldots, n$, by formula

$$(\xi, \eta)_0 = \int\limits_{\Omega_n} \xi \wedge *\eta, \ \xi, \eta \in E^q, \tag{15}$$

where $*$ is the Hodge operator, and denote the corresponding norm by $|| \cdot ||_0$. Continue scalar product (15) to direct sum $\overset{n}{\underset{q=0}{\oplus}} E^q$, such that different spaces E^q are orthogonal. Let

$\mathfrak{H}_0^q$ be a completion of space E^q by norm $||\cdot||_0$, and $P_{q\Delta}$ be an orthoprojector on $\mathfrak{H}_\Delta^q$. Introduce the scalar product on E^q by formulas

$$(\xi, \eta)_1 = (\Delta\xi, \eta)_0 + (\xi_\Delta, \eta_\Delta)_0, \tag{16}$$

$$(\xi, \eta)_2 = (\Delta\xi, \Delta\eta)_0 + (\xi, \eta)_1, \tag{17}$$

where $\omega_\Delta = P_{q\Delta}\omega$. Let $\mathfrak{H}_1^q$ and $\mathfrak{H}_2^q$ be completions of lineal E^q by corresponding norms $||\cdot||_1$ and $||\cdot||_2$, respectively. In fact, sub index means how many times q-forms are differentiable in generalized sense in the corresponding spaces. Spaces $\mathfrak{H}_l^q$, $l = 1, 2$, are Banach spaces (their Hilbert structure does not interest us further), moreover, there exist continuous and dense embeddings $\mathfrak{H}_2^q \subset \mathfrak{H}_1^q \subset \mathfrak{H}_0^q$, and for any $q = 0, 1, \ldots, n$ their exist the splittings of spaces

$$\mathfrak{H}_l^q = \mathfrak{H}_{l\Delta}^{q1} \oplus \mathfrak{H}_\Delta^q,$$

where $\mathfrak{H}_{l\Delta}^{q1} = (\mathbb{I} - P_\Delta)[\mathfrak{H}_l^q]$, $l = 0, 1, 2$.

Define spaces $\mathfrak{H}_\mathbf{2}^\mathbf{q}\mathbf{L}_2$ of smooth differential q-forms

$$w(t, x_1, x_2, ..., x_n) = \sum_{|i_1, i_2, ..., i_q| = q} \chi_{i_1, i_2, ..., i_q}(t, x_1, x_2, ..., x_n) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_n},$$

where coefficients $\chi_{i_1, i_2, ..., i_q}(t, x_1, x_2, ..., x_n) \in \mathbf{C}^1(\mathbb{R}; \mathbf{U_L L}_2)$ are stochastic $\mathbf{L}$-processes, and x_i are one-dimensional Brownian processes. We can separate time and local coordinates at non-relativistic speeds. Here time t is the same at all points of the manifold and impacts only on the coefficients of differential forms that are stochastic continuous $\mathbf{L}$-processes differentiable in the sense of Nelson–Gliklikh.

For fixed $\alpha \in \mathbb{R}, \lambda \in \mathbb{R}$ introduce operators

$$L = (\lambda + \Delta), \; M = \alpha\Delta, \tag{18}$$

where Δ is the Laplace–Beltrami operator. Consider a stochastic equation with differential forms

$$L \overset{o}{\eta} = M\eta \tag{19}$$

with Cauchy condition

$$\eta(0) = \eta_0. \tag{20}$$

Paper [8] establishes solvability of problem (19), (20). Introduce

$$\mathbf{I_L^+ L_2} = \{\eta \in \mathfrak{H}_\mathbf{2}^\mathbf{q}\mathbf{L_2} : (\cdot, \varphi_1)_\mathbf{0}\varphi_1 = \mathbf{0}, \nu_1 > \lambda\}, \tag{21}$$

and

$$\mathbf{I_L^- L_2} = \{\eta \in \mathfrak{H}_\mathbf{2}^\mathbf{q}\mathbf{L_2} : (\cdot, \varphi_1)_\mathbf{0}\varphi_1 = \mathbf{0}, \nu_1 < \lambda\}. \tag{22}$$

The following theorem is true.

Theorem 6. *For any $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$, $\eta_0 \in \mathbf{U_L L}_2$, solution $\eta = \eta(t)$ of problem (19), (20) has exponential dichotomies, and $\mathbf{I_L^+ L_2}$ and $\mathbf{I_L^- L_2}$ of form (21), (22) are infinite-dimensional stable and finite-dimensional unstable invariant spaces of equation (19), respectively.*

Remark 1. For any $\alpha \in \mathbb{R}_-$, $\lambda \in \mathbb{R}_+$ and $\eta_0 \in \mathbf{U_L L}_2$, solution $\eta = \eta(t)$ of problem (19), (20) has exponential dichotomies, and there exist finite-dimensional stable and infinite-dimensional unstable invariant spaces of equation (19). For any $\alpha \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}_-$ we can only talk about exponential stability of solutions of problem (19), (20), and the solutions of problem (19), (20) are exponentially unstable for $\alpha, \lambda \in \mathbb{R}_-$.

Conclusion

Further, we plan to continue the results of the paper in several directions. Namely, to generalize the results for the case of sectorial operator [5] and even more general case of radial operator [6]. Also, to investigate generalized Showalter–Sidorov problem, and multipoint problem [22]. Moreover, the theory of degenerate resolving groups and semigroups of operators, as well as numerical methods [23], require such a research.

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ЭКСПОНЕНЦИАЛЬНЫЕ ДИХОТОМИИ В МОДЕЛИ БАРЕНБЛАТТА – ЖЕЛТОВА – КОЧИНОЙ В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ С «ШУМАМИ»

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Исследована устойчивость решений в линейных стохастических моделях соболевского типа с относительно ограниченным оператором в пространствах гладких дифференциальных форм, определенных на гладких компактных ориентированных римановых многообразиях без края. Для этого в пространстве дифференциальных форм используем вместо обычного оператора Лапласа псевдодифференциальный оператор Лапласа – Бельтрами. В качестве начальных использованы условие Коши и условие Шоуолтера – Сидорова. В связи с недифферинцируемостью, в обычном понимании, имеющегося в модели «белого шума» используем производную стохастического процесса в смысле Нельсона – Гликлиха. Для исследования устойчивости решений устанавливаем наличие экспоненциальных дихотомий разделяющих пространство решений на устойчивое и неустойчивое инвариантные подпространства. В качестве примера используется стохастический вариант уравнения Баренблатта – Желтова – Кочиной в пространстве дифференциальных форм, определенных на гладком компактном ориентированном римановом многообразии без края.

Ключевые слова: уравнения соболевского типа; дифференциальные формы; стохастические уравнения; производная Нельсона – Гликлиха.

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