ON THE STABILITY OF TWO-DIMENSIONAL FLOWS CLOSE TO THE SHEAR

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We consider the stability problem for two-dimensional spatially periodic flows of general form, close to the shear, assuming that the ratio of the periods tends to zero, and the average of the velocity component corresponding to the "long" period is non-zero. The first terms of the long-wavelength asymptotics are found. The coefficients of the asymptotic expansions are explicitly expressed in terms of some Wronskians and integral operators of Volterra type, as in the case of shear basic flow. The structure of eigenvalues and eigenfunctions for the first terms of asymptotics is identified, a comparison with the case of shear flow is made. We study subclasses of the considered class of flows in which the general properties of the qualitative behavior of eigenvalues and eigenfunctions are found. Plots of neutral curves are constructed. The most dangerous disturbances are numerically found. Fluid particle trajectories in the self-oscillatory regime in the linear approximation are given.

Keywords: long-wave asymptotics; stability of two-dimensional viscous flows; neutral stability curves.

Introduction

Mathematical models that describe two-dimensional or spatially periodic fluid flows are widely used to study various natural phenomena in the atmosphere and the ocean [1]. In [2], a model of two-dimensional creeping motion of viscous liquid in a flat channel is considered. Based on a priori estimates, the solution of the problem is constructed and its properties are investigated. The Kolmogorov problem for a two-dimensional viscous fluid under the influence of spatially periodic external forces is considered in [3]. Using the Galerkin method, stationary and spatially periodic solutions are found numerically. In [4] nonstationary time-periodic structures are obtained using long-wave perturbations of the Kolmogorov flow.

In this paper we consider the two-dimensional ($\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$) viscous incompressible flow driven by an external forces field $\boldsymbol{F}(\boldsymbol{x}, t)$ that is periodic in x_1 and x_2 with periods ℓ_1 and ℓ_2, respectively. The flow is described by the Navier–Stokes equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} - \nu \Delta \boldsymbol{v} = -\nabla p + \boldsymbol{F}(\boldsymbol{x}, t), \quad \text{div}\, \boldsymbol{v} = 0,$$

where $\nu = 1/Re$ is the kinematic viscosity and Re is the Reynolds number. The period $\ell_1 = 2\pi$, and the ratio of the periods is characterized by the wave number α: $\ell_2 = 2\pi/\alpha$, $\alpha \to 0$. Let $\langle f \rangle$ denote the average with respect to x_1, $\langle\!\langle f \rangle\!\rangle$ denote the average over the period rectangle $\Omega = [0, \ell_1] \times [0, \ell_2]$:

$$\langle f \rangle = \frac{1}{\ell_1} \int\limits_0^{\ell_1} f(\boldsymbol{x}, t)\, dx_1, \quad \langle\!\langle f \rangle\!\rangle(t) = \frac{1}{|\Omega|} \int\limits_\Omega f(\boldsymbol{x}, t)\, dx_1\, dx_2.$$

The spatial average velocity is assumed to be given: $\langle\!\langle \boldsymbol{v} \rangle\!\rangle = \boldsymbol{q}$. The velocity field is assumed to be periodic in x_1, x_2 with the same periods ℓ_1, ℓ_2 as the field of external forces.

A longwave asymptotics ($\alpha \to 0$) is constructed for the stability problem of the steady flow close to the shear, which will be called the basic flow:

$$\boldsymbol{V} = (\alpha V_1(x_2), V_2(x_1)), \quad \langle V_2 \rangle \neq 0. \tag{1}$$

The class of flows under consideration generalizes the Kolmogorov flow with a sinusoidal velocity profile

$$\boldsymbol{V} = (0, \gamma \sin(x_1)). \tag{2}$$

The problem of investigating the stability of a two-dimensional flow under the action of spatially periodic force was proposed by A.N. Kolmogorov in his seminar. The instability of the Kolmogorov flow with respect to long-wave perturbations in the linear approximation was proved in [5]. The long-wave asymptotic behavior of the stability problem for two-dimensional parallel flows of general form

$$\boldsymbol{V} = (0, V_2(x_1)), \quad \langle V_2 \rangle \neq 0$$

was considered in [6]. Research [7] is devoted to the study of self-oscillations arising in the loss of stability of parallel flows of a viscous fluid affected by long wavelength perturbations. In [8], the main terms of the asymptotics of the secondary self-oscillatory regimes in the case of the basic flow close to parallel were found, but general rules in coefficient expressions were not obtained.

In [9] recurrence formulas for finding the k-th term of the long wavelength asymptotic for the stability of steady shear flows were derived in the case of nonzero average. The coefficients of the expansions are explicitly expressed in terms of some Wronskians, as well as integral operators of Volterra type. It is shown that the eigenvalues of the linear spectral problem are odd functions of the parameter α, and the critical viscosity is an even function. In the particular case, when the deviation of the velocity from its mean value $V_2(x) - \langle V_2 \rangle$ is an odd function of x, the coefficients of expansion of the eigenvalues in series in powers of α, starting from the third, are zero and the eigenvalues can be found exactly: $\sigma_{1,2} = \pm im\langle V \rangle \alpha, m \neq 0$. In [10], recurrence formulas for finding the kth term of the long-wave asymptotics for the stability of two-dimensional basic shear flows of a viscous incompressible fluid with zero average are derived.

The aim of this paper is to generalize the results [9] related to shear flows in the case of basic flows close to shear.

1. Equations up to α^k

Let H denote the subspace of functions $f \in L_2(0, \ell_1)$ that are orthogonal to unity: $\langle f \rangle = 0$. The operator $I : H \to H$ is the inverse of the differentiation operator and is completely continuous:

$$If = \int\limits_0^x f(s)ds - \left\langle \int\limits_0^x f(s)ds \right\rangle. \tag{3}$$

Let $W(f, g)$ denote the Wronskian of functions f and g, the auxiliary function θ characterizes the deviation of the second component of velocity from its average value:

$$W(f, g) = f\frac{dg}{dx} - g\frac{df}{dx}, \quad \frac{d^2\theta}{dx^2} = V_2 - \langle V_2 \rangle, \quad \langle \theta \rangle = 0.$$

Looking for a solution $(\widetilde{\mathbf{v}}, \widetilde{p})$ of linearized on the basic flow (1) perturbation equation in the form of normal oscillations, we obtain the linear eigenvalue problem (here and below, $x = x_1$, $z = \alpha x_2$):

$$\sigma\varphi_1 + \alpha^2\varphi_2\frac{dV_1}{dz} + \alpha V_1(z)\frac{\partial\varphi_1}{\partial x} + \alpha V_2(x)\frac{\partial\varphi_1}{\partial z} - \nu\left(\frac{\partial^2\varphi_1}{\partial x^2} + \alpha^2\frac{\partial^2\varphi_1}{\partial z^2}\right) = -\frac{\partial P}{\partial x}, \quad (4)$$

$$\sigma\varphi_2 + \varphi_1\frac{dV_2}{dx} + \alpha V_1(z)\frac{\partial\varphi_2}{\partial x} + \alpha V_2(x)\frac{\partial\varphi_2}{\partial z} - \nu\left(\frac{\partial^2\varphi_2}{\partial x^2} + \alpha^2\frac{\partial^2\varphi_2}{\partial z^2}\right) = -\alpha\frac{\partial P}{\partial z}, \quad (5)$$

$$\frac{\partial\varphi_1}{\partial x} + \alpha\frac{\partial\varphi_2}{\partial z} = 0, \quad \int_0^{2\pi}\varphi_1(x, z)dz = 0, \quad \langle\varphi_2\rangle = 0. \quad (6)$$

The value of the parameter ν at which one or several eigenvalues σ lie on the imaginary axis is called critical.

The unknown perturbations of velocity $\boldsymbol{\varphi}(x, z)$, the function $P(x, z)$, the eigenvalues σ and the critical viscosity ν are sought in the form of series in powers of α:

$$\sigma(\alpha) = \sum_{k=0}^{\infty}\sigma_k\alpha^k, \quad \nu = \nu_* + \sum_{k=1}^{\infty}\nu_k\alpha^k, \quad (7)$$

$$\boldsymbol{\varphi} = \sum_{k=0}^{\infty}\boldsymbol{\varphi}^k\alpha^k, \quad P = \sum_{k=0}^{\infty}P^k\alpha^k. \quad (8)$$

We substitute (7) – (8) in (4) – (6) and equate the coefficients with the same powers of α. Up to α^0 from the continuity equation (6) we deduce that $\varphi_1^0 = \varphi_1^0(z)$. Then it follows from (4) that $\sigma_0\varphi_1^0(z) = -\frac{\partial P^0}{\partial x}$. Whence $\sigma_0 = 0$ and $P^0 = P^0(z)$. From (5) we find the function φ_2^0, that has the same form as in the case of shear basic flow:

$$\varphi_2^0 = \frac{1}{\nu_*}\varphi_1^0(z)a_0(x), \quad a_0(x) = \frac{d\theta}{dx}. \quad (9)$$

Averaging equation (4) and equating the coefficients up to α^1, we obtain:

$$\sigma_1\varphi_1^0(z) + \langle V_2\rangle\frac{d\varphi_1^0}{dz} = 0. \quad (10)$$

From (10) we find $\sigma_1 = im\langle V_2\rangle$, $\varphi_1^0(z) = e^{-imz}$, where $m \neq 0$ is the wave number.

Up to α^k, $k \geqslant 1$, from (4) – (6) we derive the following system of equations:

$$\nu_*\frac{\partial^2\varphi_1^k}{\partial x^2} = \frac{\partial P^k}{\partial x} + \sum_{j=1}^{k}\sigma_j\varphi_1^{k-j} - \sum_{j=1}^{k-1}\nu_j\frac{\partial^2\varphi_1^{k-j}}{\partial x^2} - \sum_{j=0}^{k-2}\nu_j\frac{\partial^2\varphi_1^{k-2-j}}{\partial z^2} + $$
$$+ V_2(x)\frac{\partial\varphi_1^{k-1}}{\partial z} + \frac{dV_1}{dz}\varphi_2^{k-2} + V_1(z)\frac{\partial\varphi_1^{k-1}}{\partial x}, \quad (11)$$

$$\nu_* \frac{\partial^2 \varphi_2^k}{\partial x^2} = \sum_{j=1}^{k} \sigma_j \varphi_2^{k-j} - \sum_{j=1}^{k} \nu_j \frac{\partial^2 \varphi_2^{k-j}}{\partial x^2} - \sum_{j=0}^{k-2} \nu_j \frac{\partial^2 \varphi_2^{k-2-j}}{\partial z^2} +$$

$$+ \{W(\varphi_1^k, \theta'')\} + \langle V_2 \rangle \frac{\partial \varphi_2^{k-1}}{\partial z} + V_1(z) \frac{\partial \varphi_2^{k-1}}{\partial x} + \frac{\partial \{P^{k-1}\}}{\partial z}, \tag{12}$$

$$\frac{\partial \varphi_1^k}{\partial x} + \frac{\partial \varphi_2^{k-1}}{\partial z} = 0, \quad \int_0^{2\pi} \varphi_1^k dz = 0, \quad \langle \varphi_2^k \rangle = 0. \tag{13}$$

Here, the sum is assumed to extend over those values of j for which the upper boundary is not less than the lower one. The solvability condition for equations (11) – (12) is that the average of the right-hand side over period ℓ_1 is zero. Averaging (11) for $k \geq 2$, we get the equation for finding $\langle \varphi_1^{k-1} \rangle$:

$$\langle V_2 \rangle \left(im \langle \varphi_1^{k-1} \rangle + \frac{d \langle \varphi_1^{k-1} \rangle}{dz} \right) = \nu_{k-2} \frac{d^2 \varphi_1^0}{dz^2} - \frac{d^2}{dz^2} \langle \theta' \varphi_2^{k-2} \rangle - \sigma_k \varphi_1^0(z) -$$

$$- \sum_{j=3}^{k-1} \sigma_j \langle \varphi_1^{k-j} \rangle + \sum_{j=0}^{k-3} \nu_j \frac{d^2}{dz^2} \langle \varphi_1^{k-2-j} \rangle \equiv F_{k-1}(z). \tag{14}$$

We will call (14) the averaged equation up to α^k. On the other hand, the solvability condition of equation (14) serves to find ν_{k-2} and σ_k:

$$\langle \langle F_{k-1}(z) e^{imz} \rangle \rangle = 0. \tag{15}$$

For $k = 2$, from the solvability condition (15) for equation (14) we find: $\nu_*^2 = \langle \theta'^2 \rangle$, $\sigma_2 = 0$. For the values ν_*^2 and σ_2 found, the right-hand side of (14) is zero. To exclude trivial nonuniqueness, we put $\langle \varphi_1^1 \rangle = 0$, as in the case of shear flow [9].

The scheme for finding the k-th term of the asymptotics for $k \geq 1$ is as follows. Suppose φ_2^{k-1}, φ_1^{k-1}, P^{k-1}, σ_{k+1}, ν_{k-1}, $\langle \varphi_1^k \rangle$ are known. Applying the integral operator I defined by (3), from the continuity equation (13) we find

$$\varphi_1^k = -I \left(\frac{\partial \varphi_2^{k-1}}{\partial z} \right) + \langle \varphi_1^k \rangle. \tag{16}$$

Substituting the φ_1^k found in (11), we define P^k up to an average values. After this, from (12) we find φ_2^k. The values found in the k-th step allow writing the solvability condition for the $(k + 2)$-th term of the asymptotics. From the averaged equations up to α^{k+2} we find σ_{k+2}, ν_k, $\langle \varphi_1^{k+1} \rangle$. Then the process is repeated.

For a non-parallel basic flow (1), the coefficients of expansion of the critical viscosity ν and the eigenvalues σ have the following structure:

$$\nu_k = [\nu_k] + \widetilde{\nu_k}, \quad \sigma_{k+2} = [\sigma_{k+2}] + \widetilde{\sigma_{k+2}}, \tag{17}$$

where the square brackets are used to denote the coefficients of viscosity and eigenvalues in the case of basic shear flow, and the wave is used to denote additional term. If $V_1(z) = 0$ then that additional term is equal to zero. It will be shown later that components of

eigenfunctions and pressure φ_1^k, P^k for $k = 1, 2, 3$ and φ_2^k for $k = 1, 2$ have the same structure:

$$\varphi_1^k = [\varphi_1^k] + \widetilde{\varphi_1^k} + \langle \varphi_1^k \rangle, \tag{18}$$

$$\varphi_2^k = [\varphi_2^k] + \widetilde{\varphi_2^k}, \tag{19}$$

$$P^k = [P^k] + \widetilde{P^k} + \langle P^k \rangle, \tag{20}$$

while for large values of k, additional terms appear in the expressions of the eigenfunctions φ and the pressure P. These terms depend on the mean values of previous coefficients.

2. Asymptotic Terms up to α^1

Putting $k = 1$ in (16), taking into account the expressions of φ_2^0 in (9) and $\langle \varphi_1^1 \rangle = 0$, we find φ_1^1

$$\varphi_1^1 = -I\left(\frac{\partial \varphi_2^0}{\partial z}\right) = -\frac{1}{\nu_*}\frac{d\varphi_1^0}{dz}I(a_0),$$

and P^1, which coincides with the one found in the case of shear basic flow:

$$P^1 = q_1(x)\frac{d\varphi_1^0}{dz} + \langle P^1 \rangle, \quad q_1(x) = -2\frac{d\theta}{dx}.$$

Represent φ_2^1 as a sum (19), where $[\varphi_2^1]$ and $\widetilde{\varphi_2^1}$ satisfy the equations:

$$\nu_*\frac{\partial^2 [\varphi_2^1]}{\partial x^2} = \{W(\varphi_1^1, \theta'')\} - \nu_1\frac{\partial^2 \varphi_2^0}{\partial x^2}, \quad \nu_*\frac{\partial^2 \widetilde{\varphi_2^1}}{\partial x^2} = V_1(z)\frac{\partial \varphi_2^0}{\partial x}.$$

(Here and below the deviation of a periodic function from its period-average value is denoted by curly brackets: $\{F\} = F(x) - \langle F \rangle$). Coefficient $[\varphi_2^1]$ coincides with the coefficient φ_2^1 for the shear basic flow except for the term containing ν_1, which has not been determined yet. Suppose $a_1(\theta) = I\{W(\theta', \theta)\}$ [9], then

$$[\varphi_2^1] = \frac{1}{\nu_*^2}\frac{d\varphi_1^0}{dz}a_1(\theta) - \frac{\nu_1}{\nu_*}\varphi_2^0, \quad \widetilde{\varphi_2^1} = \frac{1}{\nu_*}V_1(z)I(\varphi_2^0) = \frac{1}{\nu_*^2}\varphi_1^0(z)V_1(z)I(a_0).$$

For $k = 3$, we transform the right-hand side of (14), taking the known values $[\nu_1] = 0$ and $[\sigma_3] = -\frac{im^3}{\nu_*^2}\langle \theta' a_1 \rangle$ into account:

$$F_2(z) = 2\widetilde{\nu}_1\frac{d^2\varphi_1^0}{dz^2} - \frac{d^2}{dz^2}\langle \theta'\widetilde{\varphi_2^1} \rangle - \widetilde{\sigma}_3\varphi_1^0(z).$$

From the solvability condition (15), taking into account the orthogonality of θ' and $\widetilde{\varphi_2^1}$, we obtain $\widetilde{\nu}_1 = 0$, $\widetilde{\sigma}_3 = 0$. As in the case of shear basic flow, $\nu_1 = 0$, $\sigma_3 = [\sigma_3]$. Hence the right-hand side of (14) is zero. To exclude trivial non-uniqueness, we put $\langle \varphi_1^2 \rangle = 0$. In the particular case when $\theta(x)$ is an odd function, $\sigma_3 = 0$, as in the case of shear basic flow.

3. Asymptotic Terms up to α^2

We derive φ_1^2 from the continuity equation (16), representing φ_1^2 as a sum (18), where

$$[\varphi_1^2] = -\frac{1}{\nu_*^2}\frac{d^2\varphi_1^0}{dz^2}I(a_1), \quad \widetilde{\varphi_1^2} = -\frac{1}{\nu_*}\frac{\partial}{\partial z}\left(V_1(z)I^2(\varphi_2^0)\right).$$

From (11) for $k = 2$, we conclude that $[P^2]$ coincides with the one found in the case of shear flow:

$$[P^2] = \frac{1}{\nu_*}\frac{d^2\varphi_1^0}{dz^2}q_2, \quad q_2 = -a_1 + I\{\theta''I(a_0)\},$$

and $\widetilde{P^2}$ satisfies the equation

$$\frac{\partial\widetilde{P^2}}{\partial x} = \nu_*\frac{\partial^2\widetilde{\varphi_1^2}}{\partial x^2} - \varphi_2^0\frac{dV_1}{dz} - V_1(z)\frac{\partial\varphi_1^1}{\partial x},$$

from which, applying the integral operator I, we find

$$\widetilde{P^2} = -2\frac{dV_1}{dz}I(\varphi_2^0) = -\frac{2}{\nu_*}\varphi_1^0(z)\frac{dV_1}{dz}I(a_0).$$

Let consider the equation (12) for $k = 2$. We represent φ_2^2 as a sum (19), where $[\varphi_2^2]$ and $\widetilde{\varphi_2^2}$ satisfy the equations

$$\nu_*\frac{\partial^2[\varphi_2^2]}{\partial x^2} = \frac{d^2\varphi_1^0}{dz^2}q_1(\theta) + \{W([\varphi_1^2],\theta'')\} - \nu_*\frac{\partial^2\varphi_2^0}{\partial z^2} - \nu_2\frac{\partial^2\varphi_2^0}{\partial x^2},$$

$$\nu_*\frac{\partial^2\widetilde{\varphi_2^2}}{\partial x^2} = \frac{1}{\nu_*}\{W(\widetilde{\varphi_1^2},\theta'')\} + \frac{\langle V_2\rangle}{\nu_*}\frac{dV_1}{dz}I(\varphi_2^0) + \frac{1}{\nu_*}V_1(z)\frac{\partial\varphi_2^1}{\partial z}. \tag{21}$$

Taking into account already known expressions of φ_2^0 and φ_1^2, we find φ_2^2:

$$[\varphi_2^2] = \frac{1}{\nu_*^3}\frac{d^2\varphi_1^0}{dz^2}a_2(\theta) - \frac{\nu_2}{\nu_*}\varphi_2^0(x,z),$$
$$a_2(\theta) = I^2[\{W(\theta'',Ia_1)\} + \nu_*^2(q_1(\theta) - a_0(\theta))]. \tag{22}$$

The coefficient $a_2(\theta)$ (22) is the same as in the case of shear basic flow; $[\varphi_2^2]$ matches structure with the corresponding expression of the shear basic flow. As it will be shown below, the coefficient ν_2 differs from the one found in the case of shear flow.

Applying the integral operator I twice, from (21) we deduce

$$\widetilde{\varphi_2^2} = \frac{1}{\nu_*}I^2\{W(\widetilde{\varphi_1^2},\theta'')\} + \frac{\langle V_2\rangle}{\nu_*}\frac{dV_1}{dz}I^3(\varphi_2^0) + \frac{1}{\nu_*}V_1(z)I(\varphi_2^1).$$

From (14) for $k = 4$ considering $[\sigma_4] = 0$, $\quad[\nu_2] = -\frac{m^2}{2\nu_*^3}\langle\theta'a_2\rangle$ we get

$$F_3(z) = 2\widetilde{\nu_2}\frac{d^2\varphi_1^0}{dz^2} - \frac{d^2}{dz^2}\langle\theta'\widetilde{\varphi_2^2}\rangle - \widetilde{\sigma_4}\varphi_1^0(z).$$

To verify the solvability condition (15), we need the expression for $\langle\langle\widetilde{\varphi_2^2 e^{imz}}\rangle\rangle$:

$$\langle\langle\widetilde{\varphi_2^2 e^{imz}}\rangle\rangle = \frac{im}{\nu_*^3}\langle\langle V_1\rangle\rangle g_2(x) + \frac{1}{\nu_*^3}\langle\langle V_1^2\rangle\rangle f_2(x), \tag{23}$$

$$g_1 = Ia_0, \quad g_2(x) = I^2\{W(Ig_1, \theta'')\} - Ia_1, \quad f_2(x) = Ig_1(x) = I^2 a_0.$$

Separating the real and imaginary parts in the solvability condition (15) for $k = 4$, we obtain

$$\widetilde{\nu}_2 = \frac{1}{2\nu_*^3}\langle\langle V_1^2\rangle\rangle\langle\theta' f_2\rangle = -\frac{1}{2\nu_*^3}\langle\langle V_1^2\rangle\rangle\langle\theta^2\rangle, \quad \widetilde{\sigma}_4 = \frac{im^3}{\nu_*^3}\langle\langle V_1\rangle\rangle\langle\theta' g_2\rangle. \tag{24}$$

Note that the imaginary part of (23) contains the mean values of the odd degrees of V_1, and the the real part contains the mean values of the even degrees of V_1.

The solution of (14) for $k = 4$ will be sought in the form $\langle\varphi_1^3\rangle = c(z)e^{-imz}$, where $c(z)$ is a 2π-periodic function. Let I_z denote the operator I defined by (3) with variable upper limit z. Condition (15) uniquely determines the function $c(z)$. Then $c(z) = e^{-imz}I_z(F_3(z)e^{imz})/\langle V_2\rangle$.

Further, in each of the orders, two special cases will be considered.

Case 1. $\theta(x)$ is an odd function (it means that the deviation $V_2 - \langle V_2\rangle$ is an odd function). Hence $g_1(x)$ is odd and from (23) it follows that $g_2(x)$ and $f_2(x)$ are even functions.

Case 2. $V_1(z)$ is an odd function of variable z. Then $\langle\langle V_1(z)\rangle\rangle = 0$ and $\widetilde{\sigma}_4 = 0$.

4. Asymptotic Terms up to α^3

From the continuity equation (13) for $k = 3$ we find φ_1^3, in the form of a sum (18), where

$$[\varphi_1^3] = -\frac{1}{\nu_*^3}\frac{d^3\varphi_1^0}{dz^3}Ia_2 - \frac{\nu_2}{\nu_*}\varphi_1^1,$$

$$\widetilde{\varphi_1^3} = -\frac{1}{\nu_*}I^3\left\{W\left(\frac{\partial\widetilde{\varphi_1^2}}{\partial z}, \theta''\right)\right\} - \frac{\langle V_2\rangle}{\nu_*^2}\frac{\partial}{\partial z}\left(\frac{dV_1}{dz}I^4\varphi_2^0\right) - \frac{1}{\nu_*}\frac{\partial}{\partial z}(V_1(z)I^2\varphi_2^1).$$

P^3 can be represented as a sum (20) where

$$[P^3] = \frac{1}{\nu_*^2}\frac{d^3\varphi_1^0}{dz^3}q_3(\theta), \quad q_3(\theta) = -a_2 + I\{\theta'' I(a_1)\} - \nu_*^2 I^2 a_0,$$

and $\widetilde{P^3}$ is determined by expression

$$\widetilde{P^3} = -I\left\{\theta''\frac{\partial\widetilde{\varphi_1^2}}{\partial z}\right\} - I^2\left\{W\left(\frac{\partial\widetilde{\varphi_1^2}}{\partial z}, \theta''\right)\right\} - 2\frac{dV_1}{dz}I(\varphi_2^1).$$

As in [10] we use the following notation. If the function f is expressed in terms of a linear combination of the functions $\varphi_1^0(z)$ and its derivatives with coefficients depending on x, then by $f(\langle\varphi_1^k\rangle)$ we denote the expression that coincides with the expression f if $\varphi_1^0(z)$ is replaced by $\langle\varphi_1^k\rangle(z)$. Since $\langle\varphi_1^3\rangle \neq 0$, it is convenient to represent φ_2^3 as a sum

$$\varphi_2^3 = [\varphi_2^3] + \widetilde{\varphi_2^3} + \varphi_2^0((\langle\varphi_1^3\rangle)), \tag{25}$$

where $[\varphi_2^3]$ has a form

$$[\varphi_2^3] = \frac{1}{\nu_*^4}\frac{d^3\varphi_1^0}{dz^3}a_3(\theta) - 2\frac{\widetilde{\nu}_2}{\nu_*}[\varphi_2^1] - \frac{\nu_3}{\nu_*}\varphi_2^0,$$

$$a_3(\theta) = I^2[\{W(\theta'', Ia_2)\} + \nu_*^2(q_2 - a_1)] - \frac{\langle\theta' a_2\rangle}{\nu_*^2}a_1(\theta) - \langle\theta' a_1\rangle I^2(a_0).$$

Then $\widetilde{\varphi_2^3}$ is determined from the following equality:

$$\widetilde{\varphi_2^3} = \frac{1}{\nu_*}I^2\left(\frac{\partial\{\widetilde{P^2}\}}{\partial z}\right) + \frac{1}{\nu_*}I^2\{W(\widetilde{\varphi_1^3}, \theta'')\} + \langle V_2\rangle I^2\left(im\widetilde{\varphi_2^2} + \frac{\partial\widetilde{\varphi_2^2}}{\partial z}\right) -$$

$$-\frac{\nu_2}{\nu_*}\widetilde{\varphi_2^1} - I^2\frac{\partial^2\widetilde{\varphi_2^1}}{\partial z^2} + \frac{1}{\nu_*}V_1(z)I(\varphi_2^2).$$

To find $\langle\varphi_1^4\rangle$, ν_3 and σ_5, we write F_4 taking into account $[\nu_3] = 0$, $\quad[\sigma_5] = \frac{im}{\nu_*^4}\langle\theta' a_3(\theta)\rangle$:

$$F_4(z) = 2\widetilde{\nu}_3\frac{d^2\varphi_1^0}{dz^2} - \frac{d^2}{dz^2}\langle\theta'\widetilde{\varphi_2^3}\rangle - \left(\widetilde{\sigma}_5 + \frac{2\widetilde{\nu}_2}{\nu_*}\sigma_3\right)\varphi_1^0(z).$$

Separating the real and imaginary parts (15) for $k = 6$, we obtain

$$\widetilde{\nu}_3 = -\frac{m^2}{2\nu_*^4}\langle\langle V_1\rangle\rangle\langle\theta' g_3\rangle, \quad \widetilde{\sigma}_5 = \frac{im^3}{\nu_*^4}\langle\langle V_1^2\rangle\rangle\langle\theta' f_3\rangle - \frac{2\widetilde{\nu}_2}{\nu_*}\sigma_3,$$

$$g_3 = I^2\{W(Ig_2, \theta'')\} + Ia_2, \quad f_3 = I^2\{W(If_2, \theta'')\} + Ig_2. \quad (26)$$

Now, unlike the previous step (24), the expression for the viscosity expansion coefficients includes the averages of the odd powers of V_1, and the eigenvalue coefficient contains the averages of even powers of V_1. The solution of (14) for $k = 5$ has the form $\langle\varphi_1^4\rangle = e^{-imz}I_z(F_4(z)e^{imz})/\langle V_2\rangle$, similar to the order α^3.

Case 1. $\theta(x)$ is an odd function. Then it follows from (26) that $f_3(x)$ and $g_3(x)$ are odd functions, hence $\nu_3 = 0$ and, taking into account $\sigma_3 = 0$, we find $\sigma_5 = [\sigma_5] = 0$.

Case 2. $V_1(z)$ is an odd function of z. Then $\langle\langle V_1(z)\rangle\rangle = 0$ and $\nu_3 = 0$, as in the case of shear basic flow.

5. Asymptotic Terms up to α^4

From the continuity equation (13) for $k = 4$ and φ_2^3 defined in (25), we find φ_1^4:

$$\varphi_1^4 = [\varphi_1^4] + \widetilde{\varphi_1^4} + \varphi_1^1(\langle\varphi_1^3\rangle) + \langle\varphi_1^4\rangle,$$

where $[\varphi_1^4]$, $\widetilde{\varphi_1^4}$ are defined by formulas:

$$[\varphi_1^4] = -\frac{1}{\nu_*^4}\frac{d^4\varphi_1^0}{dz^4}Ia_3(\theta) - 2\frac{\widetilde{\nu}_2}{\nu_*}[\varphi_1^2] - \frac{\nu_3}{\nu_*}\varphi_1^1,$$

$$\widetilde{\varphi_1^4} = \frac{2}{\nu_*}\frac{\partial^2}{\partial z^2}\left(\frac{dV_1}{dz}I^4\varphi_2^0\right) - \frac{1}{\nu_*}I^3\left\{W\left(\frac{\partial\widetilde{\varphi_1^3}}{\partial z}, \theta''\right)\right\} +$$

$$+\frac{\langle V_2\rangle}{\nu_*}I^2\left(im\widetilde{\varphi_1^3} + \frac{\partial\widetilde{\varphi_1^3}}{\partial z}\right) - \frac{\nu_2}{\nu_*}\widetilde{\varphi_1^2} - I^2\left(\frac{\partial^2\widetilde{\varphi_1^2}}{\partial z^2}\right) - \frac{1}{\nu_*}\frac{\partial}{\partial z}(V_1(z)I^2\varphi_2^2).$$

The pressure P^4 is representable as the sum $P^4 = [P^4] + \widetilde{P^4} + P^1((\langle \varphi_1^3 \rangle)) + \langle P^4 \rangle$, where

$$[P^4] = \frac{1}{\nu_*^3} \frac{d^4 \varphi_1^0}{dz^4} q_4(x) - \frac{\nu_2}{\nu_*}[P^2],$$

$$\widetilde{P^4} = -I^2 \left(\frac{\partial^2 \{\widetilde{P^2}\}}{\partial z^2} \right) - I^2 \left\{ W \left(\frac{\partial \widetilde{\varphi_1^3}}{\partial z}, \theta'' \right) \right\} - I \left\{ \theta'' \frac{\partial \widetilde{\varphi_1^3}}{\partial z} \right\} - 2 \frac{dV_1}{dz} I \varphi_2^2,$$

and q_4 is found in [9].

Representing φ_2^4 as the sum $\varphi_2^4 = [\varphi_2^4] + \widetilde{\varphi_2^4} + \varphi_2^1(\langle \varphi_1^3 \rangle) + \varphi_2^0(\langle \varphi_1^4 \rangle)$ and taking into account

$$[\varphi_2^4] = \frac{1}{\nu_*^5} \frac{d^4 \varphi_1^0}{dz^4} a_4 + \frac{\sigma_4}{\nu_*} I^2 \varphi_2^0 - \frac{\widetilde{\nu_2}}{\nu_*}[\varphi_2^2] - \frac{2\nu_3}{\nu_*}[\varphi_2^1] - \frac{2\widetilde{\nu_2}}{\nu_*^2} I^2 \{W([\varphi_1^2], \theta'')\} - \frac{\nu_4}{\nu_*} \varphi_2^0,$$

we find $\widetilde{\varphi_2^4}$:

$$\widetilde{\varphi_2^4} = \frac{1}{\nu_*} I^2 \left(\frac{\partial \widetilde{P^3}}{\partial z} \right) + \frac{\langle V_2 \rangle}{\nu_*} I^2 \left(im \widetilde{\varphi_2^3} + \frac{\partial \widetilde{\varphi_2^3}}{\partial z} \right) + \frac{1}{\nu_*} I^2 \{W(\widetilde{\varphi_1^4}, \theta'')\} +$$

$$+ \frac{\sigma_3}{\nu_*} I^2 \widetilde{\varphi_2^1} - \frac{\nu_2}{\nu_*} \widetilde{\varphi_2^2} - \frac{\nu_3}{\nu_*} \widetilde{\varphi_2^1} - I^2 \left(\frac{\partial^2 \widetilde{\varphi_2^2}}{\partial z^2} \right) + \frac{1}{\nu_*} V_1(z) I \varphi_2^3 + \frac{1}{\nu_*} I^2 a_0(x) F_3(z).$$

To find $\langle \varphi_1^5 \rangle$, σ_6 and ν_4, we write out the right-hand side of the averaged equation up to α^6:

$$F_5(z) = \nu_4 \frac{d^2 \varphi_1^0}{dz^2} - \frac{d^2}{dz^2}\langle \theta' \varphi_2^4 \rangle - \sigma_6 \varphi_1^0(z) - \sigma_3 \langle \varphi_1^3 \rangle + \nu_* \frac{d^2 \langle \varphi_1^4 \rangle}{dz^2}. \tag{27}$$

Taking into account $[\nu_4] = \frac{m^4}{2\nu_*^5}\langle \theta' a_4(\theta) \rangle, \quad [\sigma_6] = 0$, (27) yields

$$F_5(z) = 2\widetilde{\nu_4} \frac{d^2 \varphi_1^0}{dz^2} - \frac{\sigma_4}{\nu_*} \frac{d^2}{dz^2}\langle \theta' I^2 \varphi_2^0 \rangle + \frac{\widetilde{\nu_2}}{\nu_*} \frac{d^2}{dz^2}\langle \theta' [\varphi_2^2] \rangle +$$

$$+ 2\frac{\nu_3}{\nu_*} \frac{d^2}{dz^2}\langle \theta' [\varphi_2^1] \rangle + 2\frac{\widetilde{\nu_2}}{\nu_*^2} \frac{d^2}{dz^2}\langle \theta' I^2 \{W([\varphi_1^2], \theta'')\} \rangle - \frac{d^2}{dz^2}\langle \theta' \widetilde{\varphi_2^4} \rangle - \widetilde{\sigma_6} \varphi_1^0(z).$$

Separating the real and imaginary parts of (15) and considering the previous coefficients, we obtain

$$\widetilde{\nu_4} = \frac{1}{2} Re \langle \langle \langle \theta' \widetilde{\varphi_2^4} \rangle e^{imz} \rangle \rangle - \frac{\widetilde{\nu_2}}{2\nu_*}([\nu_2] - \widetilde{\nu_2}) - \frac{\widetilde{\nu_2} m^2}{\nu_*^4}\langle \theta' I^2 \{W(Ia_1, \theta'')\} \rangle,$$

$$\widetilde{\sigma_6} = m^2 Im \langle \langle \langle \theta' \widetilde{\varphi_2^4} \rangle e^{imz} \rangle \rangle - \frac{\sigma_4 m^2}{\nu_*}\langle \theta^2 \rangle - 2\frac{\nu_3 \sigma_3}{\nu_*}.$$

Introduce the notation:

$$h_1 = -I^2 \{W(f_2, \theta'')\} + \langle V_2 \rangle I f_2, \quad h_2 = -I^2 \{W(Ih_1, \theta'')\} + \langle V_2 \rangle I^2 h_1,$$

$$f_4 = I^2 \{W(If_3, \theta'')\} + I g_3, \quad g_4 = I^2 \{W(Ig_3, \theta'')\} - I a_3.$$

36

Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming & Computer Software (Bulletin SUSU MMCS), 2019, vol. 12, no. 3, pp. 28–41

Then the expression $Re\langle\langle\langle\theta'\widetilde{\varphi_2^4}\rangle e^{imz}\rangle\rangle$ has the following structure

$$Re\langle\langle\langle\theta'\widetilde{\varphi_2^4}\rangle e^{imz}\rangle\rangle = \frac{1}{\nu_*^5}\langle\langle V_1^2\rangle\rangle\langle\theta' A_1\rangle + \frac{1}{\nu_*^5}\langle\langle\left(\frac{dV_1}{dz}\right)^2\rangle\rangle\langle\theta' A_2\rangle + \frac{1}{\nu_*^5}\langle\langle V_1^4\rangle\rangle\langle\theta' I^2 f_2\rangle,$$

where A_1, A_2 are determined by the formulas

$$A_1 = -m^2 f_4 - 3\nu_2\nu_* f_2 + 2m^2 I^2 f_2, \quad A_2 = 3\nu_*^2 I^2 f_2 - I h_2.$$

Similarly, $Im\langle\langle\langle\theta'\widetilde{\varphi_2^4}\rangle e^{imz}\rangle\rangle$ has a form:

$$Im\langle\langle\langle\theta'\widetilde{\varphi_2^4}\rangle e^{imz}\rangle\rangle = \frac{m}{\nu_*^3}\langle\langle V_1\rangle\rangle\langle\theta' B_1\rangle + \frac{m}{\nu_*^5}\langle\langle V_1^3\rangle\rangle\langle\theta' B_2\rangle,$$

where B_1, B_2 are determined by the formulas:

$$B_1 = I^3\{\theta'' f_2\} + I^2\{W(f_2, \theta'')\} + I^2\{W(I^2 f_2, \theta'')\} + I^2 g_2 -$$
$$-\frac{\nu_2}{\nu_*}(2I^2\{W(Ig_1, \theta'')\} + g_2) - \frac{1}{\nu_*^2}g_4,$$
$$B_2 = I^2\{W(I^2 f_2, \theta'')\} + I f_3.$$

Note that in the order α^4, as in the order α^2, the expression for the viscosity expansion coefficients includes the averages of the even powers of V_1, and the eigenvalue coefficient contain the averages of odd powers of V_1. As in the previous order, for $k = 6$ the solution of the equation (14) has the form: $\langle\varphi_1^5\rangle = e^{-imz}I_z(F_5(z)e^{imz})/\langle V_2\rangle$.

Case 1. $\theta(x)$ is an odd function. Then $f_4(x)$ and $g_4(x)$ are even functions.

Case 2. $V_1(z)$ is an odd function of z. Then $\langle\langle V_1(z)\rangle\rangle = 0$, $\langle\langle V_1^3(z)\rangle\rangle = 0$ and $\sigma_6 = 0$.

Using the obtained formulas, we construct graphs of curves of neutral stability $y = \frac{1}{\nu(\alpha)}$. For general flows, as well as for special cases considered above, there is a similar qualitative picture. The most dangerous are perturbations at $m = 1$ (Fig. 1).
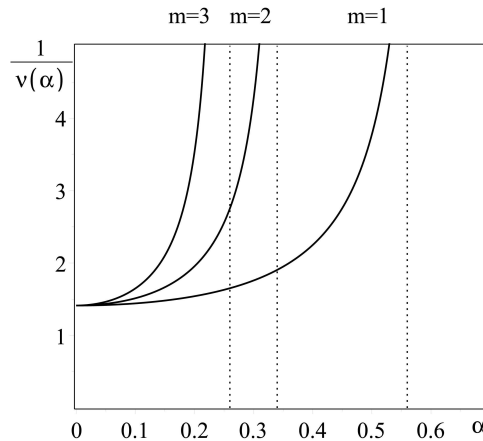
Fig. 1. Neutral stability curves

The found asymptotics allows us to investigate the trajectories of the motion of passive impurity particles in the secondary self-oscillatory flow [8]. The trajectories of particles in the linear approximation satisfy the equation

$$\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x}) + \mathbf{u}(\mathbf{x}, t), \quad \mathbf{u}(\mathbf{x}, t) = \varphi e^{i\omega t} + \varphi^* e^{-i\omega t}.$$

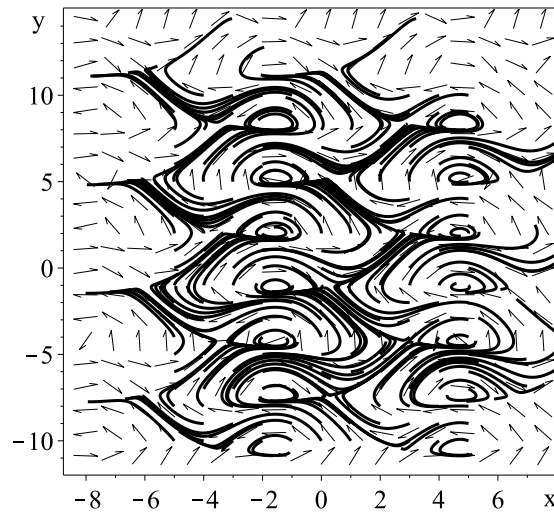The qualitative behavior of the trajectories is presented in Fig. 2.

Fig. 2. Trajectories of the motion of particles of a passive admixture

Conclusion

The first terms of the long-wavelength asymptotics with respect to the parameter α of linear spectral problem for a class of non-parallel flows close to the shear are found. A comparison with the case of the shear basic flow is made. It is shown that the expansion coefficients in terms of α eigenvalues and the critical value of viscosity have the form (17), where the square brackets are used to denote the coefficients of viscosity and eigenvalues in the case of basic shear flow, and the wave is used to denote additional term. If $V_1(z) = 0$ then that additional term is equal to zero.

The first terms of the expansion in series in the parameter α of velocity $\varphi_1^0(z)$, $\varphi_2^0(x, z)$, $\varphi_1^1(x, z)$ and pressure P^0, P^1 coincide with the coefficients for the case of shear basic flow. Average values $\langle \varphi_1^1 \rangle = \langle \varphi_1^2 \rangle = 0$, as in the case of shear basic flow, but $\langle \varphi_1^k \rangle \neq 0$ for $k \geqslant 3$.

The coefficients of the decomposition in a series of eigenfunctions have the following structure:

$$\varphi_1^k = [\varphi_1^k] + \widetilde{\varphi_1^k} + \varphi_1^1(\langle \varphi_1^{k-1} \rangle) + \langle \varphi_1^k \rangle,$$
$$\varphi_2^k = [\varphi_2^k] + \widetilde{\varphi_2^k} + \varphi_2^1(\langle \varphi_1^{k-1} \rangle) + \varphi_2^0(\langle \varphi_1^k \rangle),$$

and the coefficients of decomposition in a series of pressure have the following structure

$$P^k = [P^k] + \widetilde{P^k} + P^1(\langle \varphi_1^{k-1} \rangle) + \langle P^k \rangle.$$

Here $k = 1, 2, 3, 4$. The expressions in square brackets $[\varphi_i^k]$, $[P^k]$ are the solutions of the same equations, as in the case of shear basic flow and coincide with corresponding coefficients if $\widetilde{\nu}_j = 0$, $\widetilde{\sigma}_j = 0$.

If at least one of the conditions is met: $\theta(x)$ is odd or $V_1(z)$ is odd, then critical value of viscosity $V_1(z)$ is an even function of α as in the case of shear basic flow (up to order $k = 4$ inclusively):

$$\nu(\alpha) = \nu_* + \nu_2\alpha^2 + \nu_4\alpha^4 + O(\alpha^5), \quad \alpha \to 0.$$

If $\theta(x)$ is odd, then odd components of the eigenvalue decomposition, starting with the third, are zero:

$$\sigma(\alpha) = \sigma_1\alpha + \sigma_4\alpha^4 + \sigma_6\alpha^6 + O(\alpha^7), \quad \alpha \to 0.$$

If $V_1(z)$ is odd, then even components of the eigenvalue decomposition are zero:

$$\sigma(\alpha) = \sigma_1\alpha + \sigma_3\alpha^3 + \sigma_5\alpha^5 + O(\alpha^6), \quad \alpha \to 0.$$

Thus, under both conditions eigenvalues up to terms of order α^5 have the form

$$\sigma(\alpha) = \pm im\alpha \, \langle V_2 \rangle + O(\alpha^5), \quad \alpha \to 0,$$

as in a shear basic flow case.

For a basic flow close to a shear of general form the inequality $\nu_2 < [\nu_2]$ is fulfilled, therefore, the loss of stability of such flows occurs at higher Reynolds numbers, than for shear flow.

Neutral stability curves qualitatively coincide with the corresponding curves for the Kolmogorov-flow (2). The trajectories of the passive admixture, found numerically, consistent with Obukhov's hydrodynamics experiments [11].

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ОБ УСТОЙЧИВОСТИ ДВУМЕРНЫХ ТЕЧЕНИЙ, БЛИЗКИХ К СДВИГОВЫМ

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Рассматривается задача устойчивости двумерных пространственно-периодических течений общего вида, близких к сдвиговым, в предположении, что отношение периодов стремится к нулю, а среднее скорости вдоль «длинного» периода отлично от нуля. Найдены первые члены длинноволновой асимптотики. Коэффициенты асимптотических разложений явно выражаются через некоторые вронскианы и интегральные операторы типа Вольтерра, как и в сдвиговом случае. Выявлена структура собственных значений и собственных функций для первых членов асимптотики, произведено сравнение со сдвиговым случаем. Исследованы подклассы рассматриваемого класса течений, в которых обнаруживаются общие свойства качественного поведения собственных значений и собственных функций. Построены графики нейтральных кривых. Численно найдены наиболее опасные возмущения. Приведены траектории движения пассивной примеси во вторичном автоколебательном потоке в линейном приближении.

Ключевые слова: длинноволновая асимптотика; устойчивость двумерных течений вязкой жидкости; кривые нейтральной устойчивости.

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Вестник ЮУрГУ. Серия «Математическое моделирование
и программирование» (Вестник ЮУрГУ ММП). 2019. Т. 12, № 3. С. 28–41

41