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GENERALIZED KELLY STRATEGY

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We study the possibility of influence on the saving of allocated funds for the elimination of consequences of natural disasters. At that, we take into account statistical data on the emergence of such phenomena and the degree of actual damage. The article describes the problem of determining the optimal share of funds that either replenish or spend the principal amount according to the distribution. We prove that, under certain conditions of the distribution and a positive mathematical expectation, it is possible to choose a share that ensures the maximum possible growth of the original deposit account. At the same time, the choice of the share allows not to lose the full provision of damage recovery. This process is presented as a serial multi-stage process based on a Markov chain that takes into account only the distribution based on the statistical data of this year to plan the size of the deposit share for the next year. For simplicity, we assume that the process is established and has a constant distribution for some time. The distribution table can be changed in the case of a major change in stochastic data. We consider a serial multi-stage process of changing the monetary amount that is purposefully deposited for the renewal, replacement and restoration of security and alarm systems at burned-out facilities. The optimal stochastic control of the change in the share of the money deposit providing this restoration is carried out based on the generalized Kelly formula. An example of model validity is shown. On the basis of statistical data, the analysis of the possibility to use this model is carried out.

Keywords: probability; distribution; specialized equipment; optimal planning; model; capital; strategy.

Introduction

Let us consider the financial consequences of certain activity for the restoration of a batch of specialized security equipment or fire alarm systems caused by natural disasters, for example, fires of varying degrees. The bank processes all financial transactions. The fire occurrence with various consequences is of a random nature and this information is shown in statistics. In construction industry, there is a usage of structures that are calculated under the assumption that a highly unlikely event will not occur [1]. The paper examines changes in the initial capital in the bank. The activity of a bank or deposit is associated with the allocation of funds for the restoration of the specialized security equipment and is of a periodic nature. During this period, these funds are transferred (for example, from the state reserve) to the bank account for the specialized security equipment that survived the fire and the account balance remains positive in this case, and the funds are written off the account balance if this equipment is destroyed by fire. In case of combustion, the amount is spent on eliminating the consequences of fire, purchasing and installing new specialized security equipment. The optimization problem consists in choosing the optimal size of the share of funds, which, with a positive mathematical expectation of the distribution of fires according to the degree of damage caused, not only will not be unprofitable, but will also ensure the maximum possible increase in the accumulation of funds in the bank with an increase in the number of application periods.

Let ζ be a discrete random variable with the distribution given in Table 1. Here the values a_k can be either positive or negative. These values are associated with the costs of purchasing, installing and operating a batch of the specialized security equipment that was lost after natural disasters and the restoration of remaining ones.

Table 1

Distribution of the discrete random variable ζ

ζ	a_1	a_2	$\cdots$	a_n
P	p_1	p_2	$\cdots$	p_n

The values of top row of Table 1 depend on stochastic data calculated as a result of processing statistics on timely prevented or eliminated fires and show the percentage of saved specialized security equipment. The display of the indicated values is of stochastic nature.

Bottom row of Table 1 shows the relative frequency of occurrence of fire events according to the severity of the consequences associated with the values in top row of Table 1. If there exists no possibility to collect the "full group event" based on the empirical data collected for a certain time, we complete the table with the value p_k such that the sum of all the values in bottom row is equal to one, and the corresponding value a_k is assumed to be equal to zero.

We consider a serial multi-stage process of changing a share of the monetary amount (the planned share) for the restoration of specialized security equipment that was lost during fires and the purchase of new batches of the specialized security equipment. All payments are certainly transferred through the bank, which initially has an amount allocated by the state for these needs. We are interested in the possibility of constructing and applying an optimal algorithm for choosing a share of $0 \leqslant \zeta \leqslant 1$, on the basis of which the values a_k are determined.

1. Generalized Kelly Strategy

Let m be a cost of a set of specialized security equipment and K be an amount of funds (capital) allocated for the entire multi-stage recovery program after emergency situations ($K > m$).

The value K is gradually replenished by the city authorities over time and can also be gradually reduced due to the costs of restoring the specialized security equipment and purchasing new ones. Both the increase in the deposit and the phenomena related to inflation are not taken into account in our model. However, these processes can be accounted for by the distribution table.

Let $s = \dfrac{m}{K}$ be a planned share ($0 \leqslant s \leqslant 1$). The values K and m may change, but the share is not. Consider the classical scheme proposed by Kelly [1]. Suppose that a random variable takes two values and its distribution is shown in Table 2. The Kelly strategy (1956) [2, 3] essentially allows to determine the bank rate in percentage (share), on which one can bet taking into account all the risks and the theory of probability, in order to maximize the total average profit constantly. An example is the purchase of an option in a certain

Table 2

Binary distribution of the random variable ζ

ζ	$+b$	$-a$
P	p	$1-p$

number (a share s) of instances, the gain or loss of which is random. Solutions to this problem, if any, are determined by the Kelly formula [2, 4]

$$s = \frac{(a+b)p - a}{ab}. \tag{1}$$

Consider a random variable with n values whose distribution is shown in Table 1. The monograph [4] proposes a condition under which it is possible to determine the share of purchases ($0 \leqslant s \leqslant 1$), which makes it possible to maximize the total average profit. In order to determine the share, it is necessary to solve the equation

$$\sum_{i=1}^{n} \frac{a_i p_i}{1 + s a_i} = 0. \tag{2}$$

Note the important remark proposed by Ralph Vince [3] and devoted to the applicability of the Kelly strategy: "The Kelly formulas are applicable only to the outcomes that have Bernoulli distribution (a distribution with two possible and discrete outcomes)". In a certain sense, this statement is not true. An analysis of the application of formula (2) is given below.

2. Analysis of Possible Applications of Formula (2)

Note that, depending on the ratio of parameters and probabilities, formula (1) may not be applicable. Let $\eta = \frac{p}{a} - \frac{q}{b}$, where $q = 1 - p$. If $\eta \notin (0,1)$, then formula (1) is meaningless.

The more general formula (2) can be applied only after analyzing the situation associated with the distribution table of the random variable and the value obtained by solving equation (2). Denote the solution to (2) by s^*.

Statement 1. [4]

1) If $M[\zeta] \leqslant 0$, then $s^* = 0$, i.e. any strategy leads to ruin.

2) Let $M[\zeta] > 0$, ζ take both positive and negative values, and suppose that the solution to equation (2) s^* belongs to the interval $(0,t)$, where $t = \min\left(-\frac{1}{a_i}, a_i < 0\right)$. Then the strategy $s = s^*$ for determining the share of purchases ensures the maximum growth of the average profit.

Note that formula (1) is a special case of formula (2) for $n = 2$. Hence, condition (2) proposed in this paper can be called the generalized Kelly formula. However, for large values $n > 4$, it is difficult or impossible to obtain a solution to (2) in an explicit form, since it is necessary to solve algebraic equations of a high degree. Therefore, Ralph Vince's statement has a different meaning.

Table 3

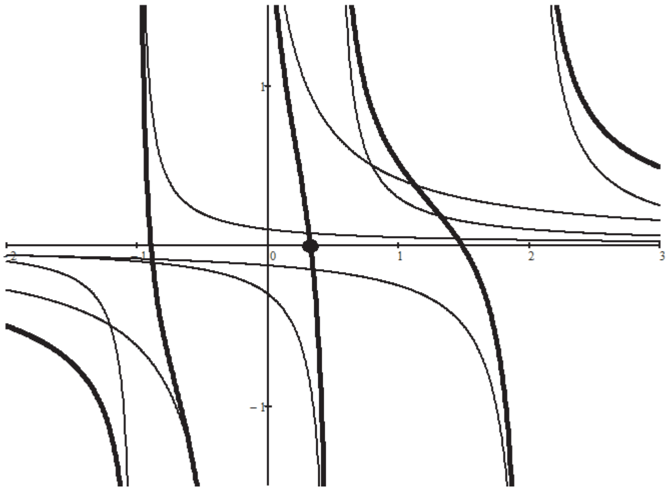
An example of the distribution of the random variable ζ

ζ	5	-2	1	$-\frac{1}{2}$
P	$0,5$	$0,15$	$0,1$	$0,25$

Consider an example that graphically illustrates the fundamental possibility to solve equation (2). Let the distribution be given by Table 3.

For this example, $M(\zeta) = 2,175 > 0$. Solution to equation $\sum\limits_{i=1}^{4} \frac{a_i p_i}{1 + s a_i} = 0$ gives the vector $(-0,89; 0,32; 1,48)$. The solution we are interested in must satisfy the condition

$s^* < t = \min\left(-\frac{1}{a_i}, a_i < 0\right)$. Therefore, $s^* \in (0; 1, 12)$. Figure shows the hyperbolic graphs of each term of the equation, as well as their sum (highlighted in bold). A satisfying solution is indicated by a dot.



Graphical solution of the equation given at the example

3. Determination of Best Money Share to Restore the Specialized Security Equipment Lost After Fires

Let Δ be an average cost allocated for the purchase and maintenance of a piece of a specialized security equipment at the facility, Ω be an average cost allocated for the replacement and maintenance of a piece of a specialized security equipment at the facility after a fire for a time period, μ be a share of sensors of a specialized security equipment destroyed by fire (we obtain the data by processing fire statistics). The values Δ, Ω are determined based on the results of the last year taking into account the development in the new year. The model applies $\Delta \cdot p_1$ instead of the entire amount. Therefore, the application of Δ is also random. The function

$$f(\mu) = \Delta(1 - \mu) + (-\Omega)\mu \tag{3}$$

determines the average cost for reconstruction of the specialized security equipment after fires. The value of function (3) can be either positive or negative, and depends on the value μ.

Let N_1 be the number of objects on which specialized security equipments are located and there were no fires during the period under consideration, therefore $\mu_1 = 0$. In the first cell of top row of the table, costs or profit are $\alpha_1 = f(\mu_1) = \Delta$. Further, $\alpha_k = f(\mu_k)$ in top row of Table 4.

Table 4

General distribution of the random variable ζ

ζ	$\alpha_1 = f(\mu_1) = \Delta$	α_2	$\cdots$	α_n
P	p_1	p_2	$\cdots$	p_n

The probability is replaced by the frequency $p_k = \frac{N_k}{\sum N_i} = \frac{N_k}{N}$. The value N corresponds to the number of all devices of a specialized security equipment in the study area on which we have statistical values of fire damage.

Let N_2 be the number of objects on which a specialized security equipment is located and there were fires that caused little damage due to timely actions of the relevant services; N_3 be the number of objects on which a specialized security equipment is located and there were fires that caused average damage; N_4 be the number of objects on which a specialized security equipment is located, there were fires that caused significant damage.

We use the following algorithm to determine μ_k. Let x_k be a direct damage calculated in thousands of rubles; y_k be material values that were saved and are calculated in thousands of rubles. In this case, $\mu_k = \dfrac{x_k}{x_k + y_k}$ determines the share of destroyed specialized security equipments located on the objects N_k. Consider Δ to be the values of the parameters characterizing the positive contribution to the bank account and Ω ($\Omega < 0$) to be the possible cost for some of the burned out equipment and the restoration of a specialized security equipment among the rest. Let Δ be equal to the cost of a fire detector, i.e. 400 rubles, the total number of production facilities be 10^9 and $\Omega = \dfrac{\Delta}{2}$.

4. Numerical Example

Consider the data on the efficiency of the fire automatics at production facilities in case of fire [1] (Table 5).

Table 5

Table to determine N_k

	Worked and completed the task	Worked and not completed the task	Not worked	Turned off
Number of fires, units	314	11	48	17
Direct damage, thousands of rubles	1291881	9897	228772	52350
Rescued material values, thousands of rubles	1139520	153500	538950	57900

Let us construct Table 6 using Table 5 and based on formula (3). We normalize top row such that its values belong to the range from -10 to 10. From bottom row of the table, we find the corresponding probabilities.

Table 6

Example of filling the table

ζ	2	$-1{,}18$	$1{,}63$	$0{,}21$	$-0{,}84$
N	0,99	$3{,}14 \cdot 10^{-3}$	$0{,}11 \cdot 10^{-3}$	$0{,}48 \cdot 10^{-3}$	$0{,}17 \cdot 10^{-3}$

The solution to the equation $\sum\limits_{i=1}^{5} \dfrac{a_i p_i}{1 + a_i s} = 0$ is the vector $s \approx (-4{,}76; -0{,}61; 0{,}84; 1{,}19)$, but only the value $s^* = 0{,}84 = \dfrac{21}{25}$ satisfies the condition $0 \leqslant s \leqslant 1$.

The share of capital that ensures the largest increase in funds on the account is $\dfrac{21}{25}$. The verification shows the fulfillment of all the conditions for the possibility of strategies with a share of capital investment, which ensure the maximum increase in savings in the process of multiple use. There are possible costs $400 \cdot \frac{21}{25} = 336$ already at the first stage, instead of the general use of the nominal value of a specialized security equipment (400 rubles). The savings are 64 rubles from each device. If we take into account the number of objects under study ($N = 10^9$) and the fact that several units of specialized security equipments are located at the object, then we get significant savings without reducing the level of security.

Undoubtedly, one can reduce costs according to several criteria using the frequency distribution table. Applying the generalized Kelly formula (2), we chose the share of capital utilization that ensures the maximum growth of the bank account over time without losing the quality of service for restoration of a specialized security equipment at the same time.

Conclusion

The paper presents a mathematical model that characterizes the payback or loss of funds during the purchase and operation of special equipment for the reporting period based on the processed stochastic material. We consider options to distribute the purchasing share of a specialized security equipment depending on the damage received during the specific placement of security equipment at various objects, the frequency of fires and the degree of material damage from fire.

In the scientific works known to the authors, there exists no practical orientation of the application of formula (2). Perhaps this is due to the authoritative but erroneous statement of Ralph Vince [3].

However, the authors are confident that the scheme described in this paper is applicable for optimization in various other areas of human activity.

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ОБОБЩЕННАЯ СТРАТЕГИЯ КЕЛЛИ

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Изучается возможность влияния на экономию выделяемых средств для ликвидации последствий стихийных бедствий с непосредственным учетом статистических данных возникновения такого рода явлений и степени реально принесенного ущерба. Рассматривается задача определения оптимальной доли размера средств, которые в зависимости от распределения или пополняют основную сумму, или ее расходуют. Показано, что при определенных условиях, накладываемых на распределение, и положительном математическом ожидании возможен выбор такой доли, которая обеспечивает максимально возможный рост первоначального депозитного вклада, не теряя полной обеспеченности реставрации повреждений. Данный процесс представляется как серийный, многоэтапный процесс марковского типа, учитывающий только распределение, построенное на статистических данных этого года для планирования величины доли вклада на следующий год. Для упрощения считается, что процесс является установившимся и некоторое время имеет постоянное распределение. При сильном изменении стохастических данных таблицу распределения можно менять. Рассматривается серийный многоэтапный процесс изменения денежной суммы, целенаправленно находящейся на депозитном вкладе для восстановления, замены и реставрации средств охраны и сигнализации на сгоревших объектах. На основе обобщенной формулы Келли проведено оптимальное стохастическое управление изменения доли вклада денежных средств, обеспечивающих данную реставрацию. Приведен пример валидности модели. На основе статистических данных проведен анализ возможности применения данной модели.

Ключевые слова: вероятность; распределение; специальные средства; оптимальное планирование; модель; капитал, стратегия.

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