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PROBABILISTIC SOLUTIONS TO THE PROBLEM OF RATIONAL CONSUMER CHOICE WITH RANDOM INCOME

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Probabilistic solutions are used when the amount of decision-makers is large. Each of them chooses the optimal solution independently of the others by solving his optimization problem. In this case, the optimal solution constructed by a randomly selected person (e.g. a consumer of goods) can be considered as a random vector. In particular, probabilistic solutions arise naturally in the rational consumer choice problem if income is assumed to be random. The problem of the utility function maximization at a time when the income of a randomly selected consumer is described as a random variable is considered as the stochastic optimization problem. The properties and distribution of the probabilistic solution of the consumer choice problem for various types of the utility function and income distribution are studied.

Keywords: stochastic optimization; probabilistic solution; the problem of consumer choice; random income; utility function.

Introduction

In the study of stochastic optimization problems, for example, problems of finding the maximum of a function that depends on a random parameter, one of the following criteria is usually used: optimization of the expected value of the objective function, maximization of the probability of reaching a certain level of the objective function (probability criterion), optimization of the fixed probability quantile. A bicriterial approach can also be used to maximize the average value and minimize the variance. In all these approaches, it is assumed that the solution to the problem is taken once and is deterministic. On the other hand, there exists a significant number of problems in which the decision is made many times, by many persons, independently of each other. In these cases, it makes sense to consider stochastic optimization problems with a random solution [1, 2]. In addition, the probabilistic solution of optimization problems naturally arises in simulation systems, when for each value of a random input a (possibly auxiliary) optimization problem is solved. The study of probabilistic solutions to optimization problems with random parameters was started in [3]. The probabilistic solutions to the problem of optimizing a function depending on a random parameter were analysed in [4]. In this article the rational consumer choice problem for a randomly selected consumer is investigated as an optimization problem with a random restriction, probabilistic properties of solutions to the optimization problem with random restrictions are studied.

1. Definition and Properties of a Probabilistic Solution

Let us consider the general probabilistic optimization problem that contains random variables in both the objective function and restrictions

$$\min_{x\in X(\xi)} f(x,\xi), \tag{1}$$

where the function $f(x, b) : R^n \times R^m \mapsto R^1$ is continuous in the aggregate of variables, the mapping $X(b) : R^m \mapsto C(R^n)$ is continuous compact-valued, $\xi = \xi(\omega)$ is a random vector defined on probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with values in R^m.

Together with probabilistic optimization problem (1), we consider the parametric optimization problem

$$\min_{x \in X(b)} f(x, b). \tag{2}$$

The set of solutions to problem (2) is denoted by

$$X^*(b) = Arg \min\{f(x, b) | x \in X(b)\}.$$

Definition 1. *A random compact set $X^*(\xi) = X^*(\xi(\omega))$ is called the probabilistic solution to stochastic programming problem* (1).

In general case, $X^*(\xi)$ is *a random compact set*. The definition and properties of random sets were considered in [5]. If problem (2) has a unique solution for any $b \in B$ then $X^*(\xi)$ is *a random vector*. Conditions of the existence of a probabilistic solution to stochastic optimization problem (1) follow from properties of the parametrical optimization problem (2).

The following Berge Maximum Theorem (see e.g. [6]) describes the conditions for the existence and continuity of a solution to the optimization problem with parameters.

Theorem 1. *Let $f : X \times B \mapsto R$ be a continuous function and $\phi : B \mapsto X$ be a compact valued and continuous correspondence, where $X \subseteq R^n$ and $B \subseteq R^m$. Then the maximum value function*

$$V(b) \triangleq \max_{x \in \phi(b)} f(x, b)$$

is well-defined and continuous and the optimal policy correspondence

$$x^*(b) \triangleq \{x \in \phi(b) | f(x, b) = V(b)\}$$

is nonempty, compact valued and upper hemicontinuous.

The following statement follows from Berge Maximum Theorem and properties of convex functions.

Lemma 1. *Let the function $f(x, b)$ be strictly concave in x and continuous in the aggregate of variables on $X \times B$, where $X \subseteq R^n$ is a convex compact set, $B \subseteq R^m$ is an open connected set. Then the solution x^* to problem* (2) *exists and is unique for all $b \in B$, and the function $X^* = \phi(b)$ is a continuous vector function of the parameter b.*

Theorem 2. *If the conditions of Lemma 1 hold and $\xi(\omega)$ is a random vector taking values from the set B then $X^*(\omega)$ is a random vector.*

Indeed, let $b = \xi(\omega)$ be a random vector. The existence and uniqueness of the solution to (2) implies that the set $X^*(\xi)$ is defined and consists of one point with probability 1. The continuity of the function $X^*(\cdot)$ implies its measurability, thus $X^*(\xi)$ is a random vector.

If the conditions of Theorem 1 are not satisfied, then the probabilistic solution to optimization problem (1) can be a random set [7].

2. Problem of Rational Consumer Behavior with Random Income

Consider the properties of probabilistic solutions to the problem of consumer choice with a random income (CCRI). A theoretical study of properties of the consumer choice problem in the case of a fixed income is given in [9]. The CCRI problem is formulated as an maximization problem with a random restriction

$$\max_{x \in X(b(\xi))} f(x), \\ X(b(\xi)) = \{x \in R^n : p^T x \leq b(\xi), \quad x \geq 0\}. \tag{3}$$

Here $f(x)$ is the utility function of goods (UF), $x = \{x_1, ..., x_n\}$ is the demand vector for various goods, $p = \{p_1, ..., p_n\}$ is the price vector for the corresponding goods, the parameter $b(\xi)$ is the income of a randomly selected consumer.

Currently, there are various approaches to choosing the best approximating income distribution function. Beta and gamma distribution, Pearson distribution, various exponential functions, etc. are used as such approximations [10]. The most common hypothesis among economists is that income distribution is subject to the logarithmically normal law [11]. In particular, the log-normal law is used to describe the structure of population incomes by official statistical services of Russia, including the Federal State Statistics Service of the Russian Federation. Another, less common theory, is the application of the Pareto distribution law to approximate income [12]. The work [13] proposes a technique for using the combined Pareto-lognormal distribution function as an approximation of the distribution of incomes of the population of Russia. The proposed approach assumes the use of the lognormal distribution function for the low- and middle-income segment, and the Pareto function for the high-income segment, since the power-law dependence better approximates the "right tail" of the distribution.

Let us study the properties of the solution to problem (3).

Denote by $X^*(\xi)$ the probabilistic solution to problem (3). Assume that the utility function $f(x)$ satisfies standard assumptions and is *neoclassical* [9], that is

- $f(x)$ is continuous and twice differentiable on the space of goods $R^n_+ = \{x \in R^n : x_i > 0, \ i = 1, \dots, n\}$;
- all first-order partial derivatives $f'_{x_i}(x)$ are positive inside R^n_+;
- the matrix of the second-order derivatives of the function $f(x)$ (Hösse matrix) is negative definite inside R^n_+.

Examples of neoclassical functions are as follows [9, 14]:

1) UF Cobb-Douglas

$$f(x) = A \prod_{i=1}^{n} x_i^{\alpha_i}, \quad 0 < \alpha_i < 1, \quad i = 1, \dots, n, \quad A > 0,$$

2) logarithmic UF

$$f(x) = \sum_{i=1}^{n} a_i \ln(1 + b_i x_i), \quad a_i > 0, \quad b_i > 0, \quad i = 1, \dots, n,$$

3) gradual additive UF

$$f(x) = \sum_{i=1}^{n} \alpha_i x_i^{\beta_i}, \quad 0 < \beta_i < 1, \quad x_i > 0, \quad \alpha_i > 0, \quad i = 1, \ldots, n,$$

4) quadratic UF

$$f(x) = \sum_{i=1}^{n} a_i x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j, \text{where} \quad a_j + \sum_{i=1}^{n} b_{ij} x_i > 0, \quad j = 1, \ldots, n.$$

If the function $f(x)$ is neoclassical, then a solution to problem (3) exists and is unique for any fixed $b > 0$ and any set of positive prices p [9].

Stone–Geary utility function [14] has the form

$$f(x) = \prod_{i=1}^{n} (x_i - a_i)^{\alpha_i}, \quad a_i > 0, \quad 0 < \alpha_i < 1, \quad i = 1, \ldots, n,$$

and satisfies the neoclassical conditions only for $x \in R_a = \{x \in R^n : \ x_i > a_i, \ i = 1, \ldots, n\}$.

The goods distribution is calculated based on the solution $X(b(\xi))$ of the optimal choice problem (3). It is natural to expect that the distribution of goods is proportional to the average expected income. But this is true only in the case of a linear dependence of the solution $X(b)$ to the optimal choice problem on the income b.

Let us consider examples of linear and nonlinear dependence of the distribution of goods on income.

Example 1. Let us take the CCRI problem with Stone–Geary utility function

$$\begin{gathered} \max \prod_{i=1}^{n} (x_i - a_i)^{\alpha_i}, \\ X(b(\xi)) = \{x \in R^n : \quad x_i \geq a_i, \quad i = 1, \ldots, n, \quad p^T x \leq b(\xi)\}. \end{gathered} \tag{4}$$

Here $a_i > 0$ is the minimum necessary amount of the i-th good that is acquired by a consumer and is not a matter of choice; the numbers $\alpha_i > 0$ characterize the value of the i-th good for a consumer.

A solution to problem (4) is unique for any set of positive prices p and any fixed $b > B_a$, where

$$B_a = \sum_{i=1}^{n} a_i p_i.$$

The optimal solution to problem (4) is written in the form:

$$x^*(b) = a + (b - B_a)c, \tag{5}$$

where $a = \{a_1, \ldots, a_n\}$, the coordinates of the vector c are determined by the relations

$$c_i = \frac{\alpha_i}{p_i A}, \quad A = \sum_{i=1}^{n} \alpha_i. \tag{6}$$

The logarithmically normal law is the most often considered as the distribution of income [11], but problem (4) has no solutions in the case of low incomes, i.e. for $b \in (0; B_a)$. Therefore, an another law is considered as the income distribution. In [12] it is proposed to apply the Pareto distribution law to approximate income. It is assumed that $b(\xi)$ is a Pareto distributed random variable with the distribution density

$$f_\xi(x) = \begin{cases} \dfrac{kb_m^k}{x^{k+1}},\ x \geq b_m, \\ 0,\ x < b_m, \end{cases} \tag{7}$$

where $b_m \geq B_a$, $k > 1$.

The probabilistic solution $X^*(\xi)$ is a random vector and its distribution is determined by the joint distribution density. It follows from equalities (5) – (6) that the functions $x_i^* = \phi_i(b)$ are linear in the parameter b. Therefore, the distribution densities $f_{\phi_i}(x)$ of the functions $x_i^* = \phi_i(b) = a_i + c_i(b - B_a)$ of a continuous random variable $b(\xi)$ can be written.

The mean values of the probabilistic solution to CCRI problem are

$$E(x_i^*(\xi)) = a_i + c_i(\bar{b} - B_a),\quad \bar{b} = E(b(\xi)) = \frac{kb_m}{k-1},\quad i = 1, \ldots, n, \tag{8}$$

where E is the mathematical expectation.

In the case of $k > 2$, the second moments of the obtained probabilistic distributions are

$$Cov(x_i^*(\xi), x_j^*(\xi)) = c_i c_j \sigma_b^2,\ \sigma_b^2 = \left(\frac{b_m}{k-1}\right)^2 \frac{k}{k-2},\quad i, j = 1, \ldots, n. \tag{9}$$

We can also calculate the required quantity of goods of each type to meet the demand with a given probability γ, i.e. the distribution quantile:

$$q_i(\gamma) : \mathcal{P}\{q :\ x_i^*(\xi) \leq q_i\} = \gamma.$$

From (5) we have that

$$q_i(\gamma) = a_i + (l_\gamma - B_a)c_i,\quad i = 1, \ldots, n, \tag{10}$$

where l_γ is a quantile of the same level γ for income distribution. In the case of Pareto distribution, the quantile l_γ is determined by the probability level and the distribution parameters b_m and k:

$$l_\gamma = \frac{b_m}{\sqrt[k]{1-\gamma}}. \tag{11}$$

Thus, in this case, the distribution of the solution (the optimal choice of goods) $x_i^*(\xi)$ may be calculated according to the density of income distribution, and the distribution of consumption of various types goods is obtained on this base.

Example 2. Let us consider the CCRI problem with the power additive utility function [14] as an example with a nonlinear dependence of the solution to the consumer choice problem on income. In this case, the solution to the optimization problem is not calculated analytically. Therefore, the simulation modelling is used to calculate the distribution of consumption.

$$\begin{gathered} f(x) = \sum_{i=1}^{n} \alpha_i x_i^{\beta_i} \to \max, \\ X = \{x \in R^n : p^T x \leq b(\xi),\quad x \geq 0\}. \end{gathered} \tag{12}$$

Here $\beta_i \in (0; 1)$, $\alpha_i > 0$ for all i.

Let the income of a randomly selected consumer $b(\xi)$ be a logarithmically normally distributed random variable with the distribution density

$$f_\xi(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right). \tag{13}$$

Although the solution to the optimal choice problem is not analytically found for an arbitrary b, it can be written as

$$X_i^* = z_i\,(y^*)^{c_i}, \quad z_i = \left(\frac{\alpha_i\beta_i}{p_i}\right)^{c_i}, \quad c_i = \frac{1}{1-\beta_i}, \tag{14}$$

where $y^* = g(b)$ is a positive root of the equation

$$G(y) = b, \quad G(y) = \sum_{i=1}^{n} (y)^{c_i}\, p_i z_i. \tag{15}$$

Since $c_i > 0$, $p_i > 0$, $z_i > 0$ for all $i = 1, \ldots, n$, then $G(z)$ is a monotonically increasing function, which is continuous on $[0, +\infty)$ and

$$G(0) = 0, \lim_{z\to+\infty} G(z) = +\infty.$$

Therefore, the positive root of equation (15) exists for all $b > 0$ and the function $y = g(b)$ is monotonically increasing.

From formula (14), the mathematical expectation of the consumers optimal choice is found as follows:

$$E(x_i^*(b(\xi))) = z_i E(g^{c_i}(b(\xi)).$$

The obtained dependence is nonlinear, therefore simulation modelling should be used to find this mathematical expectation. Since the function $g(b)$ is monotonically increasing and $c_i > 0$, $z_i > 0$, then it is enough to know a quantile of the population income distribution to find the consumption distribution quantile. The following relations are obtained

$$q_i(\gamma) = z_i Q^{c_i}(\gamma), \tag{16}$$

where $Q(\gamma)$ is the root of equation (15) for $b = l_\gamma$, i.e. $Q = g(l_\gamma)$, l_γ is the quantile of the income distribution. In the case of the lognormal distribution, the quantile is determined by the probability level and the distribution parameters μ and σ:

$$l_\gamma = \exp(\mu + \tau_\gamma\sigma), \tag{17}$$

where τ_γ is the quantile of the level γ of the standard normal distribution.

Thus, the quantile of the distribution of goods can be calculated without using simulation and it is possible to study analytically the distribution of consumers choice.

Suppose that the utility function has the form

$$f(x) = 12x_1^{0,2} + 5x_2^{0,4} + 7x_3^{0,1} + 10x_4^{0,3},$$

$$X = \{x \in R^4 : 4x_1 + 7x_2 + 5x_3 + 12x_4 \le b(\xi), \quad x \ge 0\}.$$

The optimal solution for different values of the distribution parameters μ and σ was found by numerical methods using simulation in Wolfram Mathematica. From the obtained distribution of the probabilistic solution to the problem on the optimal choice of $X^*(\xi)$ find the mathematical expectations of the components of the random vector

$$\bar{X} = E(X^*(\xi)).$$

The following results are obtained. For lognormal distribution of income with $\mu = 2$, $\sigma = 0,3$ the mathematical expectation of the random vector is $\bar{X}(\xi) = (0,765; 0,207; 0,156; 0,212)$. Fractions characterizing the components of the vector are equal $\bar{x} = (0,571; 0,155; 0,116; 0,158)$ (Fig. 1).

For $\mu = 7$, $\sigma = 0,8$ the mathematical expectation of the random vector is $\bar{X}(\xi) = (72,023; 95,727; 8,700; 39,351)$. Fractions characterizing the components of the vector are equal $\bar{x} = (0,334; 0,444; 0,040; 0,182)$ (Fig. 2).

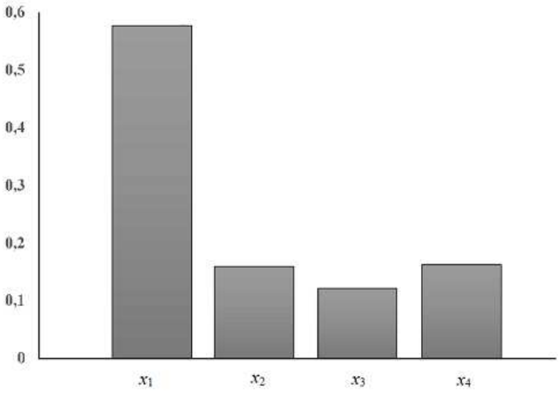

Fig. 1. $\mu = 2, \quad \sigma = 0,3$

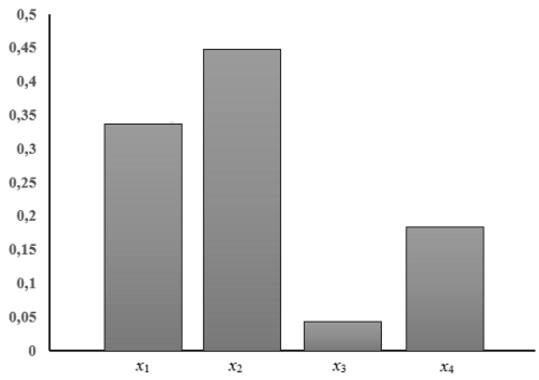

Fig. 2. $\mu = 7, \quad \sigma = 0,8$

From the considered example, it follows that the study of the demand functions and the income distribution $b(\xi)$ allows to calculate the distribution of demand for goods, and, according to the distribution of $X^*(\xi)$, can restore the average expected needs for benefits.

Conclusions

The distributions of the probabilistic solution to the stochastic optimization problem are quite diverse in the case of a nonlinear objective function. The distribution of the probabilistic solution to the rational choice problem with a random restriction is investigated.

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ВЕРОЯТНОСТНЫЕ РЕШЕНИЯ ЗАДАЧИ РАЦИОНАЛЬНОГО ПОТРЕБИТЕЛЬСКОГО ВЫБОРА СО СЛУЧАЙНЫМ ДОХОДОМ

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Вероятностные решения используются в том случае, когда количество лиц, принимающих решения, велико, каждый из них принимает оптимальное решение независимо от других, решая свою задачу оптимизации. В этом случае оптимальное решение, принятое случайно выбранным лицом (например, потребителем благ), можно рассматривать как случайный вектор. В частности, вероятностные решения естественно возникают в задаче рационального потребительского выбора в случае, когда доход предполагается случайным. Задача максимизации функции полезности в условиях, когда доход случайно выбранного потребителя описывается случайной величиной, рассматривается как задача стохастической оптимизации. Исследуются свойства и распределение вероятностного решения задачи потребительского выбора для различных видов функции полезности и распределения дохода.

Ключевые слова: стохастическая оптимизация; вероятностное решение; задача потребительского выбора; случайный доход; функция полезности.

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