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ALGORITHM FOR NUMERICAL SOLUTION OF INVERSE SPECTRAL PROBLEMS GENERATED BY STURM–LIOUVILLE OPERATORS OF AN ARBITRARY EVEN ORDER

***S.I. Kadchenko*[1], *L.S. Ryazanova*[1], *Yu.R. Dzhiganchina*[2]**
[1]Nosov Magnitogorsk State Technical University, Magnitogorsk, Russian Federation
[2]Bauman Moscow State Technical University, Moscow, Russian Federation
E-mails: sikadchenko@mail.ru, ryazanovals23@gmail.com, yulia.supn@gmail.com

The article is devoted to the construction of algorithm for solving inverse spectral problems generated by Sturm–Liouville differential operators of an arbitrary even order. The goal of solving inverse spectral problems is to recover operators from their spectral characteristics and spectral characteristics of auxiliary problems. In the scientific literature, we can not find examples of the numerical solution of inverse spectral problems for the Sturm–Liouville operator of higher than the second order. However, their solution is caused by the need to construct mathematical models of many processes arising in science and technology. Therefore, the development of computationally efficient algorithm for the numerical solution of inverse spectral problems generated by the Sturm–Liouville operators of an arbitrary even order is of great scientific interest.

In this article, we use linear formulas obtained earlier in order to find the eigenvalues of discrete semi-bounded operators and develop algorithm for solving inverse spectral problems for Sturm–Liouville operators of an arbitrary even order.

The results of the performed computational experiments show that the use of the algorithm developed in the article makes it possible to recover the values of the potentials in the Sturm–Liouville operators of any necessary even order.

Keywords: eigenvalues and eigenfunctions; discrete, self-adjoint and semi-bounded operators; Galerkin method; ill-posed problems; Fredholm integral equations of the first kind; asymptotic formulas.

Introduction

The first substantiations of linear formulas for calculating the eigenvalues of discrete semi-bounded operators were made on the basis of the method of regularized traces in the papers [1–5]. Following them, consider a discrete semi-bounded below operator T and a bounded operator P defined in a separable Hilbert space H. Let the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and the orthonormal eigenfunctions $\{v_n\}_{n=1}^{\infty}$ of the operator T be known and enumerated in the non-decreasing order of their values. Denote by ν_n the multiplicity of the eigenvalue λ_n. Let n_0 be the number of all unequal eigenvalues λ_n that belong to the inner part of the circle T_{n_0} of radius $\rho_{n_0} = \dfrac{|\lambda_{n_0+1} + \lambda_{n_0}|}{2}$ centered at the origin of the complex plane. If, for all $n \in N$, the inequalities $q_n = \dfrac{2\|P\|}{|\lambda_{n+\nu_n} - \lambda_n|} < 1$ hold, then the approximate first m_0 eigenvalues $\{\widetilde{\mu}_n\}_{n=1}^{m_0}$ of the operator $T + P$ are calculated by the formulas [2]

$$\widetilde{\mu}_n = \lambda_n + (Pv_n, v_n) + \delta_n, \ n = \overline{1, m_0}. \tag{1}$$

Moreover, the numbers δ_n satisfy the estimates

$$|\delta_n| \le (2n-1)\rho_n \frac{q^2}{1-q},\ q = \max_{n\in N} q_n,\ m_0 = \sum_{n=1}^{n_0} \nu_n.$$

Formulas (1) were obtained under the condition

$$\|P\| < 0,5|\lambda_{n+\nu_n} - \lambda_n|,\ \forall n \in N. \tag{2}$$

Further investigations of the problem on calculating the eigenvalues of discrete semi-bounded operators showed that the restrictions (2) on the norm of the operator P can be removed if the Galerkin method is used to find the eigenvalues of discrete semi-bounded below operators.

In a separable Hilbert space H, consider a discrete semi-bounded operator L with the domain $D_L \subset H$. The eigenvalues μ of the differential operator L are determined by finding solutions to the equation

$$Lu = \mu u, \tag{3}$$

satisfying homogeneous boundary conditions

$$Gu|_\Gamma = 0, \tag{4}$$

where Γ is the boundary of the domain D_L. To calculate them, we use the Galerkin method. In H, we consider the complete sequence $\{H_n\}_{n=1}^{\infty}$ of the finite-dimensional subspaces $H_n \subseteq H$. Let the systems of functions $\{\varphi_k\}_{k=1}^{n}$ be orthonormal bases of the spaces H_n. Moreover, suppose that all basis functions φ_k satisfy boundary conditions (4). Let us find an approximate solution to spectral problem (3), (4) in the form

$$u_n = \sum_{k=1}^{n} a_k(n)\varphi_k.$$

Theorem 1. *Galerkin method, which is applied to the problem on finding eigenvalues of spectral problem* (3), (4) *and constructed on the system of functions* $\{\varphi_k\}_{k=1}^{\infty}$, *converges [6].*

Theorem 2. *There exists a unique solution to the problem on finding the eigenvalues and eigenfunctions of the operator* L. *The approximate values of the eigenvalues can be found by the Galerkin method [6].*

Theorem 3. *The approximate eigenvalues* $\widetilde{\mu}_n$ *of the operator* L *are found by the formulas [7]*

$$\widetilde{\mu}_n(n) = (L\varphi_n, \varphi_n) + \widetilde{\delta}_n, \tag{5}$$

where $\widetilde{\delta}_n = \sum\limits_{k=1}^{n-1} [\widetilde{\mu}_k(n-1) - \widetilde{\mu}_k(n)]$.

From the theorems, we have

$$\lim_{n\to\infty} \widetilde{\delta}_n = 0. \tag{6}$$

Therefore, as the ordinal number of the eigenvalue calculated according to formulas (5) increases, the accuracy of its calculation increases, and this is confirmed by numerous calculations. Using the example of spectral problems generated by Sturm–Liouville operators of an arbitrary even order, which are considered in the article, it is shown

that linear formulas (5) differ from the known asymptotic formulas only by the order of errors [4]. Compared to classical methods, the formulas drastically reduce the amount of computation and allow to calculate the eigenvalues of discrete semi-bounded operators with any ordinal numbers. The formulas can be used by specialists with no special knowledge in the field of spectral operator theory.

In the articles [3, 7], algorithm for solving inverse spectral problems for discrete semi-bounded operators are developed. In this case, numerical examples were considered for differential operators of the second order. Below, using the example of Sturm–Liouville operators of an arbitrary even order, we investigate the possibility of using the developed algorithm for cases where the order of operators is greater than two.

1. Sturm–Liouville Operators of Arbitrary Even Order

Consider a method for solving inverse spectral problems based on formulas (5), using the example of problems generated by Sturm–Liouville operators of an arbitrary even order of the form

$$L_{2m}u_{2m}(s) = \mu_{2m}u_{2m}(s), \quad m \in N, \quad 0 < s < \pi, \tag{7}$$

$$u_{2m}^{(2\nu-1)}(0) = u_{2m}^{(2\nu-1)}(\pi) = 0, \quad \nu = \overline{1,m}, \tag{8}$$

where $L_{2m}u_{2m}(s) = \Big(T_{2m} + p_{2m}(s)\Big)u_{2m}(s)$, $T_{2m}u_{2m}(s) = (-1)^m \dfrac{d^{2m}u_{2m}(s)}{ds^{2m}}$. Recover the potentials $p_{2m}(s)$ by the eigenvalues μ_{2m_n} of boundary value problems (7), (8) that belong to the segments $[c_{2m}, d_{2m}]$, eigenvalues $\{\lambda_{2m_n}\}_{n=0}^{\infty}$ and eigenfunctions $\{v_{2m_n}\}_{n=0}^{\infty}$ of the corresponding unperturbed boundary value problems

$$T_{2m}v_{2m}(s) = \lambda_{2m}v_{2m}(s), \quad m \in N, \quad 0 < s < \pi, \tag{9}$$

$$v_{2m}^{(2\nu-1)}(0) = v_{2m}^{(2\nu-1)}(\pi) = 0, \quad \nu = \overline{1,m}. \tag{10}$$

In the works [8–10], it is shown that (9), (10) are self-adjoint problems, the eigenvalues λ_{2m_n} and the corresponding eigenfunctions v_{2m_n} are of the form

$$\lambda_{2m_n} = n^{2m}, \quad v_{2m_n} = \sigma_n \cos(ns), \quad \sigma_n = \begin{cases} \sqrt{\dfrac{1}{\pi}}, & n = 0, \\ \sqrt{\dfrac{2}{\pi}}, & n > 0. \end{cases}, \quad \forall m \in N. \tag{11}$$

It is known that the system of functions $\{v_{2m_n} = \sigma_n \cos(ns)\}_{n=0}^{\infty}$ is an orthonormal basis of the space $L_2[0,\pi]$. Moreover, all functions v_{2m_n} satisfy boundary conditions (10). Let us write formulas (5) for this system

$$\begin{aligned} \widetilde{\mu}_{2m_n}(n) &= (Lv_{2m_n}, v_{2m_n}) + \widetilde{\delta}_{2m_n} = \lambda_{2m_n} + (p_{2m}v_{2m_n}, v_{2m_n}) + \widetilde{\delta}_{2m_n} = \\ &= n^{2m} + \sigma_n^2 \int_0^{\pi} p_{2m}(s)\cos^2(ns)ds + \widetilde{\delta}_{2m_n}. \end{aligned} \tag{12}$$

Using (12), we construct the Fredholm integral equations of the first kind

$$Ap_{2m} \equiv \int_0^{\pi} K(x_{2m}, s)p_{2m}(s)ds = \widetilde{f}_{2m}(x_{2m}), \ x_{2m} \in [c_{2m}, d_{2m}]. \tag{13}$$

Here $K(x_{2m_n}, s) = \sigma_n^2 \cos^2(ns)$. The segments $[c_{2m}, d_{2m}]$ are chosen such that they contain the required number of known eigenvalues of spectral problems (7), (8). The exact values of the right-hand sides f_{2m} of equations (13) are unknown, but we know their approximate values $\widetilde{f}_{2m}(x_{2m_n}) = \widetilde{\mu}_{2m_n}(n) - n^{2m} - \widetilde{\delta}_{2m_n}$, for which $||f_{2m} - \widetilde{f}_{2m}||^2_{L_2[c_{2m}, d_{2m}]} \le \epsilon_{2m}$. Let the kernels $K(x_{2m}, s)$ of integral equations (13) be continuous in $\Pi_{2m} = [c_{2m}, d_{2m}] \times [0, \pi]$ and $p_{2m} \in W_2^{2m}[0, \pi]$, $\widetilde{f}_{2m}(x) \in L_2[c_{2m}, d_{2m}]$.

The problems of solving Fredholm integral equations of the first kind (13) are ill-posed. Their approximate numerical solutions $\widetilde{p}_{2m}^{\alpha}$ are found by the quadrature method, passing to finite subspaces. Let us introduce discretization grids along the s and x_{2m} axes, which are uniform along the s and non-uniform along the x_{2m} axes with an equal number N_s of nodal points.

$$s_1 = 0, \quad s_{n+1} = s_n + h_s, \quad n = \overline{1, N_s - 1}, \quad h_s = \frac{\pi}{N_s - 1},$$

$$c_{2m} = x_{2m_1} < x_{2m_2} < \ldots < x_{2m_{N_s}} = d_{2m}, \quad h_{2m_1} = h_{2m_2},$$

$$h_{2m_k} = x_{2m_k} - x_{2m_{k-1}}, \quad k = \overline{2, N_s - 1}.$$

To calculate the definite integrals in equations (13), we interpolate the functions $p_{2m}(s)$ by the Lagrange polynomials

$$p_{2m}(s) = \sum_{n=1}^{N_s} l_n(s) p_{2m_n}, \quad l_n(s) = \prod_{\substack{j=1 \\ j \ne n}}^{N_s} \frac{s - s_j}{s_n - s_j}. \tag{14}$$

Substituting (14) into integral equations (13), we find systems of linear algebraic equations for the values of the functions p_{2m} at the discretization nodes $p_{2m_n} = p_{2m}(s_n)$

$$\sum_{n=1}^{N_s} H_{kn} p_{2m_n} = \widetilde{f}_{2m}(x_{2m_k}), \tag{15}$$

where $H_{kn} = \int\limits_0^{\pi} l_n(s) K(x_{2m_k}, s) ds$, $k, n = \overline{1, N_s}$.

Let us write the constructed system of equations (15) in the matrix form

$$AP_{2m} = \widetilde{F}_{2m}. \tag{16}$$

Here A is a square matrix of the order $N_s \times N_s$, and the column matrices P_{2m}, $\widetilde{F}_{2m}$ have the form

$$A = \begin{pmatrix} H_{11} & \ldots & H_{1N_s} \\ H_{21} & \ldots & H_{2N_s} \\ \ldots & \ldots & \ldots \\ H_{N_s1} & \ldots & H_{N_sN_s} \end{pmatrix},$$

$$P_{2m} = \begin{pmatrix} p_{2m_1} \\ p_{2m_2} \\ \ldots \\ p_{2m_{N_s}} \end{pmatrix}, \quad \widetilde{F}_{2m} = \begin{pmatrix} \widetilde{f}_{2m}(x_{2m_1}) \\ \widetilde{f}_{2m}(x_{2m_2}) \\ \ldots \\ \widetilde{f}_{2m}(x_{2m_{N_s}}) \end{pmatrix}.$$

Since the condition number $cond(A)$ of the matrix A of systems of equations (16) is relatively large, and the components $\widetilde{f}_{2m}(x_{2m_k})$ of the vectors $\widetilde{F}_{2m}$ in the right-hand sides of these systems are approximate, then relative errors can lead to rather large errors of the vectors P_{2m}. In this regard, in order to partially eliminate the undesirable effects of the influence of errors, it is necessary to apply various regularization methods in order to replace the inadmissible solution vector with some "pseudosolution" vector, which is the best for the problem under consideration.

To find "pseudosolutions" to systems of linear algebraic equations (16), we use the Tikhonov regularization method with the choice of the residual regularization parameters. The regularization method of A.N. Tikhonov [12–15] is reduced to minimizing parametric functionals of the form

$$||AP_{2m} - \widetilde{F}_{2m}||^2 + \alpha_{2m}||P_{2m}||^2, \tag{17}$$

where $\alpha_{2m} > 0$ is the regularization parameter, $||P_{2m}||$ is the Euclidean norm of the vector P_{2m}. The problem of minimizing functionals (17) with respect to P_{2m} is equivalent to solving the system of linear algebraic equations

$$(A^TA + \alpha_{2m}I)\widetilde{P}_{2m}^{\alpha_{2m}} = A^T\widetilde{F}_{2m}, \tag{18}$$

in this case, the parameters α_{2m} are found by the conditions

$$||A\widetilde{P}_{2m}^{\alpha_{2m}} - \widetilde{F}_{2m}|| = E||\widetilde{F}_{2m}||. \tag{19}$$

Here $\widetilde{P}_{2m}^{\alpha_{2m}}$ are vectors of approximate solutions to integral equations (13). The superscript T in (18) means transposition of the matrix A, while E is the specified residual level and I is the identity matrix.

For fixed α_{2m}, the unique solutions $\widetilde{P}_{2m}^{\alpha_{2m}}$ to the problem of minimizing functionals (17) are explicitly expressed by the formulas:

$$\widetilde{P}_{2m}^{\alpha_{2m}} = \left(A^TA + \alpha_{2m}I\right)^{-1}A^T\widetilde{F}_{2m}. \tag{20}$$

For $\alpha_{2m} \to 0$, the solutions $\widetilde{P}_{2m}^{\alpha_{2m}}$ converge to normal solutions to systems (16) [12]. In (16), inverse matrices exist for any rank of the matrix A and any $\alpha_{2m} > 0$ [15].

Another method for finding approximate solutions to Fredholm integral equations of the first kind (13), which we use, is based on the construction of A.N. Tikhonov [12, 16] smoothing functionals of the form

$$\Phi_{\alpha_{2m}}^{2m}[p_{2m}, \widetilde{f}_{2m}] = \int_{c_{2m}}^{d_{2m}} \Big[\int_0^{\pi} K(x_{2m}, s)p_{2m}(s)ds - \widetilde{f}_{2m}(x)\Big]^2 dx_{2m} + \alpha_{2m}\Omega_{2m}[p_{2m}], \tag{21}$$

where $\Omega_{2m}[p_{2m}] = \int_0^{\pi}[p_{2m}^2(s) + qp'^2_{2m}(s)]ds$ are stabilizing functionals, $q \geq 0$.

At the discretization nodes of the domain $[0, \pi]$, the approximate values of the functions $\widetilde{p}_{2m}^{\alpha_{2m}}$ are found by the condition that functionals (21) reach the minimum values

$$\Phi_{\alpha_{2m}}^{2m}[\widetilde{p}_{2m}^{\alpha_{2m}}, \widetilde{f}_{2m}] = \inf_{p_{2m} \in W_2^{2m}[0,\pi]} \Phi_{\alpha_{2m}}^{2m}[p_{2m}, \widetilde{f}_{2m}] \tag{22}$$

under the boundary conditions

$$\widetilde{p}'^{\alpha_{2m}}_{2m}(0) = \widetilde{p}'^{\alpha_{2m}}_{2m}(\pi) = 0. \tag{23}$$

Formula (22) implies the Tikhonov equations, which, in these cases, have the form:

$$\begin{gathered}\alpha_{2m}[\widetilde{p}^{\alpha_{2m}}_{2m}(t) - q\widetilde{p}''^{\alpha_{2m}}_{2m}(t)] + \int\limits_0^\pi \Big[\int\limits_{c_{2m}}^{d_{2m}} K(x_{2m},t)K(x_{2m},s)dx_{2m}\Big]\widetilde{p}^{\alpha_{2m}}_{2m}(s)ds = \\ = \int\limits_{c_{2m}}^{d_{2m}} K(x_{2m},t)\widetilde{f}_{2m}(x_{2m})dx_{2m},\ 0 \le t \le \pi.\end{gathered} \tag{24}$$

To find the regularization parameters α_{2m} in equations (24), we use the generalized residual method, which, as applied to Fredholm integral equations of the first kind (13), is reduced to solution of the equations

$$\int\limits_{c_{2m}}^{d_{2m}} \Big[\int\limits_0^\pi K(x_{2m},s)\widetilde{p}^{\alpha_{2m}}_{2m}(s)ds - \widetilde{f}_{2m}(x_{2m})\Big]^2 dx_{2m} = 0. \tag{25}$$

In discrete form, equations (25) can be written as

$$\sum_{k=1}^{N_s}\gamma_k\sum_{n=1}^{N_s}\Big(H_{kn}\widetilde{p}^{\alpha_{2m}}_{2m_n} - \widetilde{f}_{2m_k}\Big)^2 = 0. \tag{26}$$

With this approach, instead of incorrect integral equations (13), we solve either integral equations of the second kind (24) for $q = 0$ or integro-differential equations for $q > 0$ taking into account boundary conditions (23).

Discrete versions of Tikhonov equations (24) take the form

$$\sum_{n=1}^{N_s}\Big\{\alpha_{2m}\Big[l_n(t) - ql''_n(t)\Big] + G_n(t)\Big\}\widetilde{p}^{\alpha_{2m}}_n = \widetilde{F}_{2m_k}(t),\ t \in [0,\pi]. \tag{27}$$

Here $G_n(t) = \sum\limits_{k=1}^{N_s}\gamma_k K(x_{2m_k},t)H_{kn}$, $\widetilde{F}_{2m}(t) = \sum\limits_{k=1}^{N_s}\gamma_k K(x_{2m_k},t)\widetilde{f}_{2m_k}$, γ_k are weighting factors in the quadrature formula of trapezoids with a variable step. Introducing discretization on the segment $[0,\pi]$ with an equal number N_s of nodal points

$$t_1 = 0,\quad t_{k+1} = t_k + h_s,\quad k = \overline{1,N_s-1},\quad h_s = \frac{\pi}{N_s-1},$$

we obtain a system of linear algebraic equations for finding the values of the functions $\widetilde{p}^{\alpha_{2m}}_n$ at the discretization nodes

$$\sum_{n=1}^{N_s}\Big\{\alpha_{2m}\Big[l_n(t_k) - ql''_n(t_k)\Big] + G_n(t_k)\Big\}\widetilde{p}^{\alpha_{2m}}_n = \widetilde{F}_{2m_k}(t_k),\ k = \overline{1,N_s}. \tag{28}$$

Summarizing the above, we formulate an algorithm for the numerical solution of inverse spectral problems (7), (8) using the developed method.

Algorithm. For specific values of m, consider the given N_s eigenvalues $\mu_{2m_n} \in [c_{2m}, d_{2m}]$ of spectral problems (7), (8), eigenvalues $\{\lambda_{2m_n}\}_{n=1}^{N_s}$ and eigenfunctions $\{v_{2m_n}\}_{n=1}^{N_s}$ of the corresponding perturbed problems (9), (10).

1. Construct integral equation (13).

2. Using the quadrature method, write system of linear equations (16) or (28) depending

on the applied method of recovering the values of the functions $p_{2m}^{\alpha_{2m}}(s)$ at the sampling nodes.

3. Using the generalized residual method to select the regularization parameters α_{2m} and conditions (19) or (26), find α_{2m}.

4. Using the found regularization parameter α_{2m} and formulas (20) or solving systems (28), find $\widetilde{P}_{2m}^{\alpha_{2m}}$.

5. To estimate the accuracy of the obtained approximate solution $\{\widetilde{p}_{2m}^{\alpha_{2m}}(s_n)\}_{n=1}^{N_s}$, find the average absolute error $\zeta_{2m} = \dfrac{1}{N_s}\sum\limits_{k,n=1}^{N_s}\left|H_{kn}\widetilde{p}_{2m}^{\alpha_{2m}}ds - \widetilde{f}_{2m_k}\right|$ and the residual $\xi_{2m} = \sum\limits_{k=1}^{N_s}\gamma_k\sum\limits_{n=1}^{N_s}\left(H_{kn}\widetilde{p}_{2m_n}^{\alpha_{2m}} - \widetilde{f}_{2m_k}\right)^2$ of equation (13).

2. Computational Experiment

Using the algorithm described above for solving inverse spectral problems (7), (8) generated by Sturm–Liouville operators of an arbitrary even order, in the Maple mathematical environment, computational experiments were carried out to recover the values of the potentials p_{2m} using the eigenvalues of spectral problems (7), (8) that belong to the segments $[c_{2m}, d_{2m}]$, the spectra and eigenfunctions of the corresponding unperturbed problems (9), (10). In Maple software environment, using the reserved constant Digits, the operations with real numbers can be performed with a given mantissa, which allows to find numerical solutions to Fredholm integral equations of the first kind (13). All the calculation results given below were obtained with Digits=541.

Table 1

Reconstructed values of potentials at discretization nodes for Sturm–Liouville operators of various even orders

n	s_n	$p(s_n)$	$\widetilde{p}_2^{\alpha_2}(s_n)$	$\widetilde{p}_4^{\alpha_4}(s_n)$	$\widetilde{p}_6^{\alpha_6}(s_n)$	$\widetilde{p}_{16}^{\alpha_{16}}(s_n)$	$\widetilde{p}_{32}^{\alpha_{32}}(s_n)$
1	0,0000	2,0000	1,8647	1,8596	1,8596	1,8596	1,8596
2	0,2244	3,1724	4,0700	4,0645	4,0645	4,0645	4,0645
3	0,4488	4,4454	5,8664	5,8630	5,8630	5,8630	5,8630
4	0,6732	5,8192	7,3732	7,3737	7,3737	7,3737	7,3737
5	0,8976	7,2937	8,6902	8,6954	8,6954	8,6954	8,6954
6	1,1220	8,8689	9,9029	9,9124	9,9123	9,9123	9,9123
7	1,3464	10,5448	11,0864	11,0986	11,0986	11,0986	11,0986
8	1,5708	12,3214	12,3087	12,3214	12,3214	12,3214	12,3214
9	1,7952	14,1987	13,6343	13,6450	13,6449	13,6449	13,6449
10	2,0196	16,1767	15,1269	15,1333	15,1333	15,1333	15,1333
11	2,2440	18,2555	16,8533	16,8538	16,8538	16,8538	16,8538
12	2,4684	20,4349	18,8866	18,8804	18,8404	18,8804	18,8804
13	2,6928	22,7151	21,3099	21,2974	21,2975	21,2975	21,2975
14	2,9172	24,0960	24,2211	24,2038	24,2039	24,2039	24,2039
15	3,1416	27,5776	27,7382	27,7179	27,7180	27,7180	27,7180
ζ_{2m}			$1,48\cdot10^{-27}$	$3,08\cdot10^{-31}$	$1,14\cdot10^{-34}$	$1,10\cdot10^{-32}$	$3,35\cdot10^{-79}$
ξ_{2m}			$3,71\cdot10^{-51}$	$7,63\cdot10^{-55}$	$3,67\cdot10^{-58}$	$3,68\cdot10^{-77}$	$4,24\cdot10^{-103}$

To check the developed algorithm, using Tikhonov equations (24), the values of the same potentials $p_{2m}(s) = s^2+5s+2$ were recovered at the discretization nodes for different orders $2m$ ($m = 1, 2, 3, 4, 8$) of the differential operators in inverse spectral problems (7), (8). Some of the results of these calculations are shown in Table 1.

In this case, the number of eigenvalues of spectral problems (7), (8) that belong to the segment $[c_{2m}, d_{2m}]$ used in the algorithm must coincide with the number of discretization nodes of the segment $[0, \pi]$ in which the values of the potentials $\widetilde{p}_{2m}^{\alpha_{2m}}$ of spectral problems (7), (8) are found. The calculations were carried out using systems (28).

The calculation results given in Table 1 show that the developed algorithm for solving inverse spectral problems (7), (8) are computationally efficient and allow to recover the values of the potentials p_{2m} for different orders of differential operators with good accuracy.

The algorithm developed above assume that any segments $[c_{2m}, d_{2m}]$ containing their eigenvalues can be taken to recover the potentials p_{2m} in problems (7), (8). Table 2 shows the results of calculations that allow to restore the values at discretization nodes using different segments $[c_6, d_6]$ for the potential $p_6 = \sin(s)$ of spectral problem (7), (8) with $m = 3$. The calculations were carried out using formulas (20).

Table 2

The reconstructed values of the potential at the nodes of discretization using various segments containing the known eigenvalues of problem (7), (8)

n	s_n	$p_6(s_n)$	$\widetilde{p}_6^{\alpha_6}(s_n)$ $c_6 = 7,296184 \cdot 10^2$ $d_6 = 2,413757 \cdot 10^7$ $\alpha_6 = 1,985200 \cdot 10^{-34}$	$\widetilde{p}_6^{\alpha_6}(s_n)$ $c_6 = 6,252350 \cdot 10^{10}$ $d_6 = 2,084224 \cdot 10^{11}$ $\alpha_6 = 2,288800 \cdot 10^{-75}$	$\widetilde{p}_6^{\alpha_6}(s_n)$ $c_6 = 2,081952 \cdot 10^{12}$ $d_6 = 4,195873 \cdot 10^{12}$ $\alpha_6 = 1,500000 \cdot 10^{-85}$
1	0,000000	0,000000	0,000021	0,000026	0,000026
2	0,224399	0,222521	0,222540	0,222544	0,222544
3	0,448799	0,433884	0,433897	0,433900	0,433900
4	0,673198	0,623490	0,623494	0,623496	0,623496
5	0,897598	0,781831	0,781827	0,781826	0,781826
6	1,121997	0,900969	0,900956	0,900953	0,900953
7	1,346397	0,974928	0,974909	0,974905	0,974905
8	1,570796	1,000000	0,999979	0,999974	0,999974
9	1,795196	0,974928	0,974909	0,974905	0,974905
10	2,019595	0,900969	0,900956	0,900953	0,900953
11	2,243995	0,781831	0,781827	0,781826	0,781826
12	2,468394	0,623490	0,623494	0,623496	0,623496
13	2,692794	0,433884	0,433897	0,433900	0,433900
14	2,917193	0,222521	0,222540	0,222544	0,222544
15	3,141593	0,000000	0,000021	0,000026	0,000026
ζ_6			$6,518733 \cdot 10^{-21}$	$4,160036 \cdot 10^{-43}$	$8,241057 \cdot 10^{-49}$
ξ_6			$1,055714 \cdot 10^{-33}$	$3,581863 \cdot 10^{-74}$	$2,001959 \cdot 10^{-84}$

The results of calculations showed that the recovering approximate values of the potential $p_6 = \sin(s)$ using the developed algorithm practically does not depend on the choice of the segments $[c_6, d_6]$.

Conclusion

Using the example of spectral problems generated by Sturm–Liouville operators of an arbitrary even order (7), (8), we developed efficient computational algorithm that allow to find the values of the potentials $p_{2m}(s)$ by the eigenvalues μ_{2m_n} of these problems that belong to the segments $[c_{2m}, d_{2m}]$, eigenvalues $\{\lambda_{2m_n}\}_{n=0}^{\infty}$ and eigenfunctions $\{v_{2m_n}\}_{n=0}^{\infty}$ of the corresponding unperturbed boundary value problems (9), (10).

The results of the performed computational experiments confirm the correctness of the idea to develop a new method for solving inverse spectral problems (7), (8) based on the

use of formulas (5). It should be noted that the algorithm for recovering the values of the potentials are quite simple and computationally efficient in their numerical implementation for any m. The order of the differential operators is $2m$ and, in inverse spectral problems (7), (8), does not lead to additional computational difficulties when finding their numerical solutions.

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АЛГОРИТМЫ ЧИСЛЕННОГО РЕШЕНИЯ ОБРАТНЫХ СПЕКТРАЛЬНЫХ ЗАДАЧ, ПОРОЖДЕННЫХ ОПЕРАТОРАМИ ШТУРМА – ЛИУВИЛЛЯ ПРОИЗВОЛЬНОГО ЧЕТНОГО ПОРЯДКА

С.И. Кадченко[1]***, Л.С. Рязанова***[1]***, Ю.Р. Джиганчина***[2]

[1]Магнитогорский государственный технический университет им. Г.И. Носова, г. Магнитогорск, Российская Федерация

[2]Московский государственный технический университет имени Н.Э. Баумана, г. Москва, Российская Федерация

Статья посвящена построению алгоритмов решения обратных спектральных задач, порожденных дифференциальными операторами Штурма – Лиувилля произвольного четного порядка. Целью решения обратных спектральных задач является восстановление операторов по их спектральным характеристикам и спектральным характеристикам вспомогательных задач. В научной литературе примеров численного решения обратных спектральных задач для оператора Штурма-Лиувилля выше второго порядка, мы не встречали, хотя их решение вызвано необходимостью построения математических моделей многих процессов возникающих в науке и технике. Поэтому разработка вычислительно эффективных алгоритмов численного решения обратных спектральных задач, порожденных операторами Штурма – Лиувилля произвольного четного порядка, представляет большой научный интерес.

В статье, используя линейные формулы, полученные ранее, для нахождения собственных значений дискретных полуограниченных операторов, разработаны алгоритмы решения обратных спектральных задач для операторов Штурма – Лиувилля произвольного четного порядка.

Результаты проведенных вычислительных экспериментов показали, что используя разработанные в статье алгоритмы можно восстанавливать значения потенциалов в операторах Штурма – Лиувилля любого необходимого четного порядка.

Ключевые слова: собственные значения и собственные функции; дискретные, самосопряженные и полуограниченные операторы; метод Галеркина; некорректно поставленные задачи; интегральные уравнения Фредгольма первого рода; асимптотические формулы.

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Сергей Иванович Кадченко, доктор физико-математических наук, профессор, профессор кафедры «Прикладная математика и информатика», Магнитогорский государственный технический университет имени Г.И. Носова (г. Магнитогорск, Российская Федерация), sikadchenko@mail.ru.

Любовь Сергеевна Рязанова, кандидат педагогических наук, доцент, доцент кафедры «Прикладная математика и информатика», Магнитогорский государственный технический университет имени Г.И. Носова (г. Магнитогорск, Российская Федерация), ryazanovals23@gmail.com.

Юлия Рустамовна Джиганчина, студентка, Московский государственный технический университет имени Н.Э. Баумана (г. Москва, Российская Федерация), yulia.supn@gmail.com.

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