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THE PROBLEM OF IDENTIFYING THE TRAJECTORY OF A MOBILE POINT SOURCE IN THE CONVECTIVE TRANSPORT EQUATION

Kh.M. Gamzaev, Azerbaijan State Oil and Industry University, Baku, Azerbaijan, xan.h@rambler.ru

We consider the problem of identifying the trajectory of a mobile point source described by the Delta function in a one-dimensional linear convective transport equation under a given additional boundary condition. To solve this problem, the Delta function is approximated by a continuous function and a discrete analog of the problem is constructed using finite-difference approximations in the form of an implicit difference scheme. To solve the resulting difference problem, we propose a special representation that allows to split the problem into two mutually independent linear first-order difference problems at each discrete value of a time variable. The result is an explicit formula for determining the position of a mobile point source for each discrete value of a time variable. Based on the proposed computational algorithm, numerical experiments were performed for model problems.

Keywords: convective transport equation; mobile point source; identification problem; source motion law; delta function approximation.

Introduction

It is known that the process of one-dimensional transfer of a substance or any physical quantity (mass, momentum, energy, etc.) by a medium moving at a speed $\nu(x,t)$, with disregard of diffusion and in the presence of sources (drains) in the medium, is described by the convective transfer

$$\frac{\partial u}{\partial t} + \nu(x,t)\frac{\partial u}{\partial x} + \lambda(x,t)u = g(x,t),$$

where the term $\lambda(x,t)u(x,t)$ describes the absorption or release of a substance and the term $g(x,t)$ describes the action of an external source. This equation is used to describe a wide class of processes in ecology, heat transfer, hydrodynamics, acoustics, plasma physics, etc. [1–3]. Numerous papers are devoted to analytical and numerical research of direct initial boundary value problems for the convective transport equation [1–5].

It should be noted that in many practical cases, external sources are represented as a mobile point source and the Dirac Delta function is used to describe such sources, i.e.

$$g(x,t) = q(t)\delta(x - r(t)),$$

where $\delta(x - r(t))$ is the Dirac Delta function, $r(t)$ is the law of motion of a point source, and $q(t)$ is the power of the source. One of the main problems that arise in the study of processes with mobile point sources is to determine the law of motion of the source, i.e. the trajectory of the source. However, at present, the problems associated with identifying the trajectory of mobile point sources are not sufficiently studied. In this paper, the problem of identifying the trajectory of a mobile point source is presented as an inverse problem of mathematical physics.

1. Problem Statement and Solution Method

Let us consider the convective transport equation in the presence of a mobile point source

$$\frac{\partial u}{\partial t} + \nu(x,t)\frac{\partial u}{\partial x} + \lambda(x,t)u = q(t)\delta(x - r(t)),\ 0 < x \le l,\ 0 < t \le T, \tag{1}$$

under the following conditions

$$u(x,0) = \varphi(x),\ 0 \le x \le l, \tag{2}$$

$$u(0,t) = \theta(t),\ 0 \le t \le T. \tag{3}$$

It is known that the direct problem for equation (1) consists of defining a function from equation (1) with the given coefficients $\nu(x,t)$, $\lambda(x,t)$, the right-hand side $q(t)\delta(x - r(t))$, and additional conditions (2), (3). Let us assume that in addition to the unknown function $u(x,t)$, the trajectory of a mobile point source $r(t)$ is also unknown, and we need to construct this function using the following additional condition

$$u(l,t) = f(t), 0 \le t \le T, \tag{4}$$

where $f(t)$ is the specified function.

Thus, the identification problem is to determine the functions $u(x,t)$ and $r(t)$ that satisfy equation (1) and conditions (2) – (4). This problem belongs to the class of inverse problems related to the recovery of the right parts of partial differential equations [6,7]. To eliminate the singularity in equation (1), we approximate the Delta function with a continuous function [8]. For this purpose, we use the following ratio

$$\delta(x - r(t)) = \sqrt{\varepsilon/\pi}e^{-\varepsilon(x-r(t))^2},$$

where ε is a positive number. Having previously de-dimensioned the spatial variable x by introducing a scale l^{-1}, we take into account the approximation of the Delta function and conditions (2) – (4) in order to represent equation (1) as follows

$$\frac{\partial u}{\partial t} + \frac{\nu(x,t)}{l}\frac{\partial u}{\partial x} + \lambda(x,t)u = q(t)\frac{\sqrt{\varepsilon/\pi}}{l}e^{-\varepsilon(x-\xi(t))^2},\ 0 < x < 1,\ 0 < t \le T, \tag{5}$$

$$u(x,0) = \varphi(x),\ 0 \le x \le 1, \tag{6}$$

$$u(0,t) = \theta(t),\ 0 \le t \le T, \tag{7}$$

$$u(1,t) = f(t),\ 0 \le t \le T, \tag{8}$$

where $\xi(t) = r(t)l^{-1}$.

First, we construct a discrete analog of problem (5) – (8). To this end, we introduce a uniform difference grid in a rectangular domain $\{0 \le x \le 1,\ \ 0 \le t \le T\}$

$$\overline{\omega} = \{(t_j, x_i) : x_i = i\Delta x,\ \ t_j = j\Delta t,\ \ i = 0,\ 1,\ 2, ..., n,\ \ j = 0,\ 1,\ 2,\ ..., m\}$$

with the step $\Delta x = 1/n$ for the variable x and the step $\Delta t = T/m$ for the time t.

Using implicit time approximation, the discrete analog of problem (5) – (8) on the difference grid $\overline{\omega}$ is represented as

$$\frac{u_i^j - u_i^{j-1}}{\Delta t} + \frac{v_i^j}{l}\frac{u_i^j - u_{i-1}^j}{\Delta x} + \lambda_i^j u_i^j = \frac{q^j\sqrt{\varepsilon/\pi}}{l}e^{-\varepsilon(x_i-\xi^j)^2},\ i = 1, 2, \ldots, n-1, \tag{9}$$

$$u_0^j = \theta^j, \tag{10}$$

$$u_n^j = f^j, j = 1, 2, \ldots, m, \tag{11}$$

$$u_i^0 = \varphi(x_i), i = 0, 1, 2, \ldots, n, \tag{12}$$

where $u_i^j \approx u(x_i, t_j)$, $\xi^j \approx \xi(t_j)$, $\lambda_i^j = \lambda(x_i, t_j)$, $v_i^j = v(x_i, t_j)$, $q^j = q(t_j)$, $\theta^j = \theta(t_j)$, $f^j = f(t_j)$.

The constructed difference problem (9) – (12) is a system of linear algebraic equations in which the approximate values of the desired functions $u(x,t)$ and $\xi(t)$ in the nodes of the difference grid $\overline{\omega}$ are unknown, i.e. u_i^j, ξ^j, $i = 0, 1, 2, \ldots, n,$, $j = 1, 2, \ldots, m.$

In order to divide difference problem (9) – (12) into mutually independent subproblems, each of which can be solved independently, we take

$$e^{-\varepsilon(x_{i-1}-\xi^j)^2} \approx e^{-\varepsilon(x_i-\xi^j)^2}$$

and the solution to problem (9)–(12) for each fixed value j, $j = 1,\ 2,\ \ldots, m$, we represent as [9–11]

$$u_i^j = w_i^j + p_i^j e^{-\varepsilon(x_i-\xi^j)^2}, i = 0,\ 1,\ 2,\ \ldots, n, \tag{13}$$

where w_i^j, p_i^j are variables, which are not yet known.

Substituting the expression u_i^j in each equation of system (9), (10), we get

$$\left[\frac{w_i^j - u_i^{j-1}}{\Delta t} + \frac{v_i^j}{l}\frac{w_i^j - w_{i-1}^j}{\Delta x} + \lambda_i^j w_i^j\right] +$$

$$+e^{-\varepsilon(x_i-\xi^j)^2}\left[\frac{p_i^j}{\Delta t} + \frac{v_i^j}{l}\frac{p_i^j - p_{i-1}^j}{\Delta x} + \lambda_i^j p_i^j - \frac{q^j\sqrt{\varepsilon/\pi}}{l}\right] = 0,$$

$$w_0^j + p_0^j e^{-\varepsilon(x_i-\xi^j)^2} = \theta^j.$$

From the last relations we obtain the following first order difference problems for determining auxiliary variables w_i^j, p_i^j

$$\frac{w_i^j - u_i^{j-1}}{\Delta t} + \frac{v_i^j}{l}\frac{w_i^j - w_{i-1}^j}{\Delta x} + \lambda_i^j w_i^j = 0,\ i = 1,\ 2,\ ...,\ n, \tag{14}$$

$$w_0^j = \theta^j, \tag{15}$$

$$\frac{p_i^j}{\Delta t} + \frac{v_i^j}{l}\frac{p_i^j - p_{i-1}^j}{\Delta x} + \lambda_i^j p_i^j = \frac{q^j\sqrt{\varepsilon/\pi}}{l}, i = 1,\ 2,\ ...,\ n, \tag{16}$$

$$p_0^j = 0,\ j = 1, 2, 3, ..., m. \tag{17}$$

It is obvious that the solutions to obtained difference problems (14), (15) and (16), (17) for each fixed value $j = 1,\ 2,\ \dots, m$, regardless of ξ^j, can be determined by the formulas

$$w_i^j = \frac{l\Delta x}{l\Delta x + \nu_i^j \Delta t + \lambda_i^j l \Delta x \Delta t} u_i^{j-1} +$$

$$+ \frac{\nu_i^j \Delta t}{l\Delta x + \nu_i^j \Delta t + \lambda_i^j l \Delta x \Delta t} w_{i-1}^j,\ i = \overline{1, n}, w_0^j = \theta^j, \tag{18}$$

$$p_i^j = \frac{\nu_i^j \Delta t}{l\Delta x + \nu_i^j \Delta t + \lambda_i^j l \Delta x \Delta t} p_{i-1}^j + \frac{q^j \sqrt{\varepsilon/\pi} \Delta x \Delta t}{l\Delta x + \nu_i^j \Delta t + \lambda_i^j l \Delta x \Delta t},\ i = \overline{1, n},\ p_0^j = 0. \tag{19}$$

And substituting representation (13) in (11), we have

$$w_n^j + p_n^j e^{-\varepsilon (x_n - \xi^j)^2} = f^j.$$

From here, we can determine the approximate value of the desired function $\xi(t)$ for $t = t_j$, i.e. ξ^j

$$\xi^j = x_n - \sqrt{-\frac{1}{\varepsilon} \ln \left| \frac{f^j - w_n^j}{p_n^j} \right|}. \tag{20}$$

Thus, the computational algorithm for solving difference problem (9) – (12) by determining u_i^j, $i = \overline{0, n}$, and ξ^j for each fixed value $j = 1,\ 2,\ \dots, m$ consists of the following steps.

Step 1. Solve two independent first-order difference problems (14), (15) and (16), (17) with respect to auxiliary variables w_i^j, p_i^j, $i = \overline{0, n}$, using formulas (18) and (19).

Step 2. Determine the approximate value of the desired function $\xi(t)$ for $t = t_j$, i.e. ξ^j, by formula (20).

Step 3. Calculate the values of variables u_i^j using formula (13).

2. Results of Numerical Calculations

To find out the effectiveness of the proposed computational algorithm, a numerical experiment was conducted for model problems with dimensionless variables. The numerical experiment was carried out according to the following scheme.

1. For a given function $\xi(t)$, $0 \le t \le T$, the solution to problem (5) – (7) is defined, i.e. the function $u(x, t)$, $0 \le x \le 1$, $0 \le t \le T$.

2. The found dependency $f(t) = u(1, t)$ is accepted as accurate data for solving the inverse recovery problem $\xi(t)$.

Table presents the results of the numerical experiment conducted for the case $l =$10000, $T =$200, $\varepsilon =$12,57, $\lambda(x, t) =$0,002, $v(x, t) =\ 2$, $\theta(t) = 2$, $\varphi(x) = 0$. Here t is the time, ξ^t and $\overline{\xi}$ are the exact and calculated values of the function $\xi(t)$, respectively. As recoverable functions, we use $\xi(t) = 0,6 + 0,3 \sin 4t$, $\xi(t) = t/T$, $\xi(t) = -4t^2/T^2 + 4t/T$.

Table

Results of numerical calculations

t_j	$\xi(t) = 0,6 + 0,3\sin 4t$		$\xi(t) = t/T$		$\xi(t) = -4t^2/T^2 + 4t/T$	
	ξ^t	$\overline{\xi}$	ξ^t	$\overline{\xi}$	ξ^t	$\overline{\xi}$
10	0,824	0,824	0,050	0,050	0,190	0,190
20	0,302	0,302	0,100	0,100	0,360	0,360
30	0,774	0,774	0,150	0,150	0,510	0,510
40	0,666	0,666	0,200	0,200	0,640	0,640
50	0,338	0,338	0,250	0,250	0,750	0,750
60	0,884	0,884	0,300	0,300	0,840	0,840
70	0,484	0,484	0,350	0,350	0,910	0,910
80	0,472	0,472	0,400	0,400	0,960	0,960
90	0,888	0,888	0,450	0,450	0,990	0,990
100	0,345	0,345	0,500	0,500	1,000	0,997
110	0,653	0,653	0,550	0,550	0,990	0,989
120	0,785	0,785	0,600	0,600	0,960	0,960
130	0,301	0,301	0,650	0,650	0,910	0,910
140	0,814	0,815	0,700	0,700	0,840	0,840
150	0,613	0,613	0,750	0,750	0,750	0,750
160	0,368	0,368	0,800	0,800	0,640	0,640
170	0,896	0,897	0,850	0,850	0,510	0,510
180	0,437	0,437	0,900	0,900	0,360	0,358
190	0,521	0,522	0,950	0,950	0,190	0,191
200	0,868	0,868	1,00	0,998	0,000	0,490

Table 1 shows that, in all three cases, the values of the desired function are restored with high accuracy. In this case, the relative error of restoring the function values does not exceed 0,07%, 0,22% and 0,3%, except at the point $x = 0$, for the first, second and third case, respectively. The results of numerical calculations show that when a point source approaches the observation point $x = 1$, the accuracy of identifying source coordinate increases.

Analysis of the results of the numerical experiment shows that the proposed computational algorithm can be used in the study of a wide class of processes with a mobile point source.

Conclusion

The identification problem for a one-dimensional linear convective transport equation related to the restoration of the trajectory of a mobile point source is considered. The computational algorithm for solving this problem is based on the approximation of the Delta function, the discretization of the problem, and the use of a special representation for solving the difference problem. The proposed method allows to consistently determine the coordinates of a mobile point source and the distribution of the substance in the considered area in each time layer.

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ЗАДАЧА ИДЕНТИФИКАЦИИ ТРАЕКТОРИИ ПОДВИЖНОГО ТОЧЕЧНОГО ИСТОЧНИКА В УРАВНЕНИИ КОНВЕКТИВНОГО ПЕРЕНОСА

Х.М. Гамзаев, Азербайджанский государственный университет нефти и промышленности, г. Баку, Азербайджан

Рассматривается задача идентификации траектории подвижного точечного источника, описываемого дельта функцией, в одномерном линейном уравнении конвективного переноса по заданному дополнительному граничному условию. Для решения рассматриваемой задачи сначала дельта функция аппроксимируется непрерывной функцией и строится дискретный аналог задачи с помощью конечно-разностных аппроксимаций в виде неявной разностной схемы. Для решения полученной разностной задачи предлагается специальное представление, позволяющее на каждом дискретном значении временной переменной расщепить задачу на две взаимно независимые линейные разностные задачи первого порядка. В результате получена явная формула для определения положения подвижного точечного источника при каждом дискретном значении временной переменной. На основе предложенного вычислительного алгоритма были проведены численные эксперименты для модельных задач.

Ключевые слова: уравнение конвективного переноса; подвижный точечный источник; задача идентификации; закон движения источника; аппроксимация дельта функции.

Ханлар Мехвали оглу Гамзаев, доктор технических наук, профессор, кафедра «Общая и прикладная математика», Азербайджанский государственный университет нефти и промышленности (г. Баку, Азербайджан), xan.h@rambler.ru.

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