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INVARIANT MANIFOLDS OF SEMILINEAR SOBOLEV TYPE EQUATIONS

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The article is devoted to a review of the author's results in studying the stability of semilinear Sobolev type equations with a relatively bounded operator. We consider the initial-boundary value problems for the Hoff equation, for the Oskolkov equation of nonlinear fluid filtration, for the Oskolkov equation of plane-parallel fluid flow, for the Benjamin–Bon–Mahoney equation. Under an appropriate choice of function spaces, these problems can be considered as special cases of the Cauchy problem for a semilinear Sobolev type equation. When studying stability, we use phase space methods based on the theory of degenerate (semi)groups of operators and apply a generalization of the classical Hadamard–Perron theorem. We show the existence of stable and unstable invariant manifolds modeled by stable and unstable invariant spaces of the linear part of the Sobolev type equations in the case when the phase space is simple and the relative spectrum and the imaginary axis do not have common points.

Keywords: Sobolev type equations; invariant manifolds; Oskolkov equations; Hoff equation; Benjamin–Bon–Mahoney equation.

Dedicated to the 70th anniversary of the Teacher
Professor Georgy Anatolyevich Sviridyuk

Introduction

Currently, a fairly large number of models of engineering and natural science are described by problems for equations (systems of equations) in which the operator at the time derivative is not invertible. Some classes of these equations can be considered as linear

$$L\dot{u} = Mu, \tag{1}$$

semilinear

$$L\dot{u} = Mu + N(u) \tag{2}$$

and nonlinear

$$L\dot{u} = F(u) \tag{3}$$

Sobolev type equations. Here all the operators $L,\ M,\ N,\ F$ are defined in Banach spaces, $L,\ M$ are linear operators, while $N,\ F$ are nonlinear operators, and $\ker L \neq \{0\}$.

The Cauchy problem

$$u(0) = u_0 \tag{4}$$

for equations (1), (2) and (3) may not have a solution, and if a solution exists, then the solution may not be unique. Therefore, various approaches were developed to study these equations. We prefer the phase space method, the foundations of which were laid down in [24] and continued in [2–5] and others.

At present, Sobolev type equations are studied in various aspects. For example, the papers [6–8] are devoted to optimal control problems for Sobolev type equations, while the solvability of a multipoint initial-final value problem for a Sobolev-type equation is considered in [39], and Sobolev type equations in spaces of differential forms are studied in [17]. In the works [11–14], high-order Sobolev type equations are studied. The solvability of the Cauchy problem and the Showalter–Sidorov problem for a linear Sobolev type equation in spaces of "noises" was studied in [1] in the case of (L, p)-sectoriality of the operator M and in [2] in the case of (L, p)-radiality of the operator M. A stochastic linear Sobolev type equation on a manifold is considered in [17, 18] and in quasi-Banach spaces – in [38]. In the paper [3], a multipoint initial-finite value problem for a stochastic Sobolev type equation is studied. The paper [23] is devoted to dynamical measurements in spaces of "noises".

This article is of a survey nature and contains results on the local stability of semilinear Sobolev type equations, which we formulate using the concepts of stable and unstable invariant manifolds. In the study, first of all, we use the phase space method. Here we define the set of initial values u_0 for which there exists a unique local solution to problem (1), (4) or a solution to problem (2), (4). Then it is assumed that the given set (called the phase space) is a simple smooth manifold in a neighborhood of some point u_0. In this case, by virtue of the Cauchy theorem, problem (2), (4) (and, as a special case, problem (1), (4)) has a unique solution. Second, due to the assumptions about the simplicity of the phase space, we transfer the results of the classical Hadamard–Perron theorem to equation (2).

In this paper, we review the results, which are a continuation of the results on the stability of equations of the form (2) in the case of (L, p)-boundedness of the operator M, see [30]. The article consists of Introduction, five sections and References. Section 1 considers the construction of projectors, the splitting of spaces, and the actions of operators on these spaces; in addition, conditions for the existence of invariant spaces of equation (1) and invariant manifolds of equation (2) are indicated. The next four sections are devoted to applications of the results of Section 1. Namely, results on the existence of stable and unstable invariant manifolds of the Hoff equation are presented in Section 2; of the Oskolkov equation for nonlinear filtration – in Section 3; of the Oskolkov equation for a plane-parallel fluid flow – in Section 4; of the Benjamin–Bona–Mahoney equation — in Section 5.

1. Invariant Manifolds of Sobolev Type Equations

Consider the operators $L,\ M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, where $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces. By a *L-resolvent set of the operator* M we mean the set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$, while by a *L-spectrum of the operator* M we mean $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$. If the set $\sigma^L(M)$ is bounded, then the operator M is said to be a (L, σ)-*bounded operator.*

Let the operator M be (L, σ)-bonded. Construct the projectors

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} (\mu L - M)^{-1} L d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} L(\mu L - M)^{-1} d\mu \in \mathcal{L}(\mathfrak{F}),$$

which split the spaces $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ and $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$, where $\mathfrak{U}^0$ $(\mathfrak{U}^1) = \ker P$ $(\mathrm{im} P)$, $\mathfrak{F}^0$ $(\mathfrak{F}^1) = \ker Q$ $(\mathrm{im} Q)$, while the contour $\gamma \subset \mathbb{C}$ bounds a domain containing $\sigma^L(M)$. Denote by L_k (M_k) the restriction of L (M) on $\mathfrak{U}^k$, $k = 0, 1$. By virtue of the splitting theorem (see, for example, [28]), the operators L_k $(M_k) \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, there exist the operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$.

A vector-function $u \in C^k((-\tau,\tau);\mathfrak{U})$, $k \in \mathbb{N}\cup\{\infty\}$ satisfying equation (2) for some $\tau \in \mathbb{R}_+$ is said to be a *solution* to this *equation*. A solution $u = u(t)$ to equation (2) is said to be a *solution to problem* (2), (4), if equality (4) holds for some $u_0 \in \mathfrak{U}$.

Definition 1. [28] *The set $\mathfrak{P} \in \mathfrak{U}$ is said to be the phase space of equation (2), if*

(i) any solution $u = u(t)$ to equation (2) belongs to $\mathfrak{P}$, i.e. $u(t) \in \mathfrak{P}$ for each $t \in (-\tau,\tau)$;

(ii) for any $u_0 \in \mathfrak{P}$, there exists a unique solution $u \in C^k((-\tau,\tau);\mathfrak{U})$, $k \in \mathbb{N}\cup\{\infty\}$ to problem (2), (4).

Denote $H = L_0^{-1}M_0 \in \mathcal{L}(\mathfrak{U}^0)$ and $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$. The operator M is said to be (L,p)-*bounded* operator, if M is (L,σ)-bounded operator and $H \equiv \mathbb{O},\ p = 0$ or $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$. Consider the set

$$\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(Mu + N(u)) = 0\}.$$

Theorem 1. [28] *Let the operator M be (L,p)-bounded, $p \in \{0\}\cup\mathbb{N}$, the operator $N \in C^k(\mathfrak{U},\mathfrak{F})$, and the set $\mathfrak{M}$ be a simple Banach C^l-manifold at the point u_0. Then, for some $\tau \in \mathbb{R}_+$, there exists a unique solution $u \in C^m((-\tau,\tau);\mathfrak{M})$, $m = \min\{k,l\}$, to equation (2) passing through the point u_0.*

Definition 2. [30] *A subspace $\mathfrak{P} \subset \mathfrak{B}$ is called an invariant space of equation (1), if, for any $u_0 \in \mathfrak{P}$, a solution $u = u(t)$ to problem (1), (4) belongs to $\mathfrak{P}$, i.e. $u(t) \in \mathfrak{P}$ for any $t \in \mathbb{R}$.*

Definition 3. [30] *Let $\mathfrak{U}^1$ be a phase space, and $\mathfrak{U}^{1k}$, $k = 1,2$, be invariant spaces of equation (1), where $\mathfrak{U}^1 = \mathfrak{U}^{11} \oplus \mathfrak{U}^{12}$. We say that the solutions $u = u(t)$ to equation (1) have an exponential dichotomy (e-dichotomy) if*

(i) $\|u^1(t)\|_{\mathfrak{U}} \leq N_1 e^{-\nu_1(s-t)}\|u^1(s)\|_{\mathfrak{U}} \qquad s \geq t, \quad \nu_1 > 0,$

(ii) $\|u^2(t)\|_{\mathfrak{U}} \leq N_2 e^{-\nu_2(t-s)}\|u^2(s)\|_{\mathfrak{U}} \qquad t \geq s, \quad \nu_2 > 0,$

where $u^k \in \mathfrak{U}^{1k}, \quad k = 1,2.$

Theorem 2. [30] *Let the operator M be (L,p)-bounded, $p \in \{0\}\cup\mathbb{N}$ and $\sigma^L(M)\bigcap\{i\mathbb{R}\} = \emptyset$. Then the solutions $u = u(t)$ to equation (1) have an exponential dichotomy.*

Suppose that the following condition holds:

$$\left.\begin{array}{c} \sigma^L(M) = \sigma^L_+(M)\bigcup\sigma^L_-(M),\ \textit{and} \\ \sigma^L_{+(-)}(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\mu > (<)0\},\ \sigma^L_{+(-)}(M) \neq \oslash \end{array}\right\}. \tag{5}$$

Then we can construct the projectors

$$P_{l(r)} = \frac{1}{2\pi i}\int\limits_{\gamma_{l(r)}} R^L_\mu(M)d\mu,\ Q_{l(r)} = \frac{1}{2\pi i}\int\limits_{\gamma_{l(r)}} L^L_\mu(M)d\mu,$$

where the contour $\gamma_{l(r)}$ belongs to the left (right) half-plane and bounds a domain containing the part of the L-spectrum of the operator M that belongs to the given half-plane.

Definition 4. [35] *By a stable (unstable) invariant manifold of the equation (2) we mean the set*

$$\mathfrak{M}^{+(-)} = \{u_0 \in \mathfrak{U} : \|P_{l(r)}u_0\|_{\mathfrak{U}} \le R_1,\ \|u(t,u_0)\|_{\mathfrak{U}} \le R_2,\ t \in \mathbb{R}_{+(-)}\}$$

such that

(i) $\mathfrak{M}^{+(-)}$ *is diffeomorphic to a closed ball in* $\mathfrak{I}^{+(-)}$*;*

(ii) $\mathfrak{M}^{+(-)}$ *touches* $\mathfrak{I}^{+(-)}$ *at the origin;*

(iii) for any $u_0 \in \mathfrak{M}^{+(-)}$ *and for* $t \to +(-)\infty$, $\|u(t,u_0)\|_{\mathfrak{U}} \to 0$.

Theorem 3. [35] *Let the operator* M *be* (L,p)*-bounded,* $p \in \{0\} \cup \mathbb{N}$*, condition (5) be satisfied, and the operator* $N \in C^\infty(\mathfrak{U},\mathfrak{F})$ *be such that* $N(0) = 0$, $N'_0 = \mathbb{O}$*. Then, for some* $R_j,\ j = 1,2$*, there exist stable and unstable invariant manifolds of equation (2). Moreover, if, for some* $u_0 \in \mathfrak{M}$*, we have* $\|P_{l(r)}u_0\|_{\mathfrak{U}} \le R_1$ *and* $\|u(t,u_0)\|_{\mathfrak{U}} \le R_2$ *for* $t \to +(-)\infty$*, then* $u_0 \in \mathfrak{M}^{+(-)}$.

2. Invariant Manifolds of Hoff Equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of the class C^∞. In the cylinder $\Omega \times \mathbb{R}$, consider the Hoff equation

$$(\lambda + \Delta)y_t = \alpha y + \beta y^3, \tag{6}$$

which models the buckling dynamics of an I-beam in the case of $n = 1$ [4]. The desired function $y = y(x,t)$, $(x,t) \in \Omega \times \mathbb{R}$, has physical meaning of the deviation of the beam from the vertical, the parameter $\lambda \in \mathbb{R}_+$ characterizes the load, and the parameters α, $\beta \in \mathbb{R}$ characterize the material properties.

Reduce equation (6) to equation (2). Let $\mathfrak{U} = L_4$, $\mathfrak{F} = W_2^{-1}$ (hereinafter, all function spaces are defined on the domain Ω). Define the operators L, M and N by the formulas

$$\langle Lu, v\rangle = \int\limits_\Omega (\lambda uv - u_{x_k}v_{x_k})dx \quad \forall u,v \in \overset{\circ}{W}{}_2^1,$$

$$\langle Mu, v\rangle = \alpha \int\limits_\Omega uvdx, \quad \langle N(u), v\rangle = \beta \int\limits_\Omega u^3vdx \quad \forall u,v \in L_4,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in L_2. For $n \le 4$, the operators $L,\ M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ due to the continuity and density of the embedding $\overset{\circ}{W}{}_2^1 \hookrightarrow L_4$ and continuity of the embedding $L_4 \hookrightarrow (L_4)^* \cong L_{\frac{4}{3}} \hookrightarrow W_2^{-1}$.

Lemma 1. [34] *(i) Let* $n \le 4$*, then, for any* $\lambda \in \mathbb{R}_+$, $\alpha \in \mathbb{R} \setminus \{0\}$*, the operator* M *is* $(L,0)$*-bounded;*

(ii) Let $n \le 4$*, then, for any* $\beta \in \mathbb{R}$*, the operator* $N \in C^\infty(\mathfrak{U};\mathfrak{F})$*, where* $N(0) = 0$ *and* $N'_0 \equiv \mathbb{O}$.

Theorem 4. [34] *(i) Let* $-\lambda \notin \{\lambda_k\}$, $\alpha\beta > 0$ *and* $n \le 4$*. Then the phase space of equation (6) coincides with the space* $\mathfrak{U}$.

(ii) Let $-\lambda \in \{\lambda_k\}$, $\alpha\beta > 0$ *and* $n \le 4$*. Then the phase space of equation (6) is a simple Banach* C^∞*-manifold*

$$\mathfrak{M} = \{u \in \mathfrak{U} : \int\limits_\Omega (\alpha + \beta u^2)u\varphi_l dx = 0, \quad \lambda_l = -\lambda\}$$

modeled by the subspace $\mathfrak{U}^1 = \{u \in \mathfrak{U} : \langle u, \varphi_l \rangle = 0, \lambda_l = -\lambda\}$.

The continuation of these results is the work [4], in which the question of the stability of equation (6) is studied. The paper [36] studies equation (6) on graphs, [18] deals with equation (6) on manifolds, and [11] — in spaces of random **K**-values.

Theorem 5. [4] *Let* $n \le 4,\ \alpha,\ \beta,\ \lambda \in \mathbb{R}_+$. *Then if*

(i) $\lambda \le -\lambda_1$, *then, in a neighborhood of the zero point, equation* (6) *has only a stable invariant manifold, which coincides with* $\mathfrak{M}$;

(ii) $-\lambda_1 < \lambda$, *then, in a neighborhood of the zero point, equation* (6) *has the finite-dimensional unstable invariant manifold* $\mathfrak{M}^u$, $\dim \mathfrak{M}^u = \max\{\lambda_l : -\lambda_l \le \lambda\}$, *and the infinite-dimensional stable invariant manifold* $\mathfrak{M}^s$, $\operatorname{codim} \mathfrak{M}^s = \dim\ \mathfrak{M}^u + \dim\ \ker L$.

3. Invariant Manifolds of Oskolkov Equation of Non-Linear Filtration

The non-classical equation

$$(\mathbb{I} - æ\Delta)g_t = \nu\Delta g - |g|^{p-2}g,\ p \ge 2, \tag{7}$$

describes dynamics of the pressure of an incompressible viscoelastic fluid filtering in a porous medium [13]. The parameters æ, ν characterize elastic and viscous fluid properties, respectively. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of the class C^∞. Reduce equation (7) defined in the cylinder $\Omega \times \mathbb{R}$ to semilinear Sobolev type equation (2) defined in the Banach spaces $\mathfrak{U}$ and $\mathfrak{F}$. Let $\mathfrak{U} = \overset{\circ}{W}{}^1_2$, $\mathfrak{F} = W_2^{-1}$. Define the operators L, M and N by the formulas

$$< Lu, v >= \int\limits_\Omega (uv + æu_{x_i}v_{x_i})dx,\ < Mu, v >= -\nu \int\limits_\Omega u_{x_i}v_{x_i}dx,$$

$$< N(u), v >= -\int\limits_\Omega |u|^{p-2}uvdx,\ u, v \in \mathfrak{U}.$$

For $n \ge 3$, $2 \le p \le 4/(n-2) + 2$, the operators $L,\ M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, the operator $N : \mathfrak{U} \to \mathfrak{F}$.

Lemma 2. [32] *(i) For* $n \ge 3$ *and* $2 \le p \le 4/(n-2)+2$, *the operator* M *is* $(L, 0)$*-bounded;*
(ii) the operator $N \in C^1(\mathfrak{U}; \mathfrak{F})$, $N(0) = 0,\ N'_0 \equiv \mathbb{O}$.

Consider the set $\mathfrak{M} \subset \mathfrak{U}$ of the form

$$\mathfrak{M} = \left\{ \begin{array}{l} \mathfrak{U},\ \text{ if } æ^{-1} \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} :< Mu + N(u), \varphi_l >= 0\},\ \lambda_l = æ^{-1} \end{array} \right\}$$

and the space $\mathfrak{U}^1$ of the form

$$\mathfrak{U}^1 = \left\{ \begin{array}{l} \mathfrak{U},\ \text{ if } æ^{-1} \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} :< u, \varphi_l >= 0\},\ \lambda_l = æ^{-1} \end{array} \right\}.$$

Theorem 6. [32] *For any* $æ \in \mathbb{R} \setminus \{0\},\ \nu \in \mathbb{R}_+,\ n \ge 3,\ 2 \le p \le 4/(n-2) + 2$, *the phase space of equation* (7) *is the set* $\mathfrak{M}$, *which is a simple Banach* C^1*-manifold modeled by the space* $\mathfrak{U}^1$.

Theorem 7. [5] *For any* $æ \in \mathbb{R}_-$, $\nu \in \mathbb{R}_+$, $n \geq 3$, $2 \leq p \leq 4/(n-2)+2$, *in a neighborhood of the zero point, equation* (7) *has at most a finite-dimensional stable invariant manifold* $\mathfrak{M}^s$, $\dim \mathfrak{M}^s = \max\{l : \lambda_l^{-1} < æ\}$, *and an infinite-dimensional unstable invariant manifold* $\mathfrak{M}^u$, $\mathrm{codim}\mathfrak{M}^u = \dim \mathfrak{M}^s + \dim \ker L$.

4. Invariant Manifolds of Oskolkov Equation of Plane-Parallel Flow

Let $\Omega \in \mathbb{R}^2_{(x_1,x_2)}$ be a bounded domain with the boundary $\partial\Omega$ of the class C^∞. In the cylinder $\Omega \times \mathbb{R}$, consider the Oskolkov equation

$$(\lambda - \Delta)\Delta\psi_t = \nu\Delta^2\psi - \frac{\partial(\psi, \Delta\psi)}{\partial(x_1, x_2)}, \tag{8}$$

which models a plane-parallel flow of a viscoelastic incompressible fluid [14].

In order to reduce equation (8) to equation (2), we set

$$\mathfrak{U} = \{u \in W_2^4 : u(x_1, x_2) = \Delta u(x_1, x_2) = 0,\ (x_1, x_2) \in \partial\Omega\},\ \mathfrak{F} = L_2(\Omega),$$

and define the operators L, M and N by the formulas

$$L : u \to (\lambda - \Delta)\Delta u,\ M : u \to \nu\Delta^2 u,$$

$$N : u \to -\frac{\partial(u, \Delta u)}{\partial(x_1, x_2)}.$$

By construction, $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$.

Lemma 3. [29] *(i) For* $n \geq 3$ *and* $2 \leq p \leq 4/(n-2)+2$, *the operator* M *is* $(L, 0)$*-bounded;*
(ii) the operator $N \in C^1(\mathfrak{U}; \mathfrak{F})$, $N(0) = 0$, $N'_0 \equiv \mathbb{O}$.

Theorem 8. [29] *For any* $\lambda \in \mathbb{R}$, $\nu \in \mathbb{R} \setminus \{0\}$, *the phase space of equation* (8) *is the set*

$$\mathfrak{M} = \begin{cases} \mathfrak{U},\ \lambda \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} : \langle Mu + N(u), \varphi_l\rangle = 0,\ \lambda = \lambda_l\}, \end{cases}$$

which is a simple Banach C^∞*-manifold modeled by the space*

$$\mathfrak{U}^1 = \begin{cases} \mathfrak{U},\ \lambda \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} : \langle u, \varphi_l\rangle = 0,\ \lambda = \lambda_l\}. \end{cases}$$

Theorem 9. *For any* $\lambda \in \mathbb{R}$, $\nu \in \mathbb{R}_+$, *in a neighborhood of the zero point, equation* (8) *has a finite-dimensional unstable invariant manifold* $\mathfrak{M}^u$, $\dim \mathfrak{M}^u = \max\{l : \lambda_l > \lambda\}$, *and an infinite-dimensional stable invariant manifold* $\mathfrak{M}^s$, $\mathrm{codim}\mathfrak{M}^s = \dim \mathfrak{M}^u + \dim \ker L$.

5. Invariant Manifolds of Benjamin–Bona–Mahoney Equation

The equation

$$\lambda z_t - z_{xxt} = \nu z_{xx} - z z_x \tag{9}$$

models long waves in dissipative and dispersive media [15]. In order to reduce equation (9) to equation (2), we set

$$\mathfrak{U} = \{u \in W_p^{l+2}(-\pi,\pi) : u(-\pi) = u(\pi) = 0\},\ \mathfrak{F} = W_p^l(-\pi,\pi),$$

where $l \in \{0\} \cup \mathbb{N},\ p \in [2,+\infty)$. Define the operators $L,\ M,\ N : \mathfrak{U} \to \mathfrak{F}$ by the formulas

$$L = \lambda - \frac{\partial^2}{\partial^2},\ M = \nu\frac{\partial^2}{\partial^2},\ N : u \to -u_x u.$$

Since the embeddings $\mathfrak{U} \hookrightarrow \mathfrak{F}$ are continuous, the operators $L,\ M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$.

Lemma 4. [33] *(i) For any $\lambda,\ \nu \in \mathbb{R}\setminus\{0\}$, the operator M is $(L,0)$-bounded.*
(ii) For any $\nu,\ \lambda \in \mathbb{R}\setminus\{0\}$, the operator $N \in C^\infty(\mathfrak{U};\mathfrak{F})$, where $N(0) = 0$ and $N'_0 \equiv \mathbb{O}$.

Construct the set $\mathfrak{M}$ and the space $\mathfrak{U}^1$. In this case, they have the form

$$\mathfrak{M} = \begin{cases} \mathfrak{U},\ \lambda \neq -n^2; \\ \{u \in \mathfrak{U} : \int\limits_{-\pi}^{\pi} (\nu u_{xx} - u_x u)\sin lx dx = 0,\ \lambda = l^2\}, \end{cases}$$

$$\mathfrak{U}^1 = \begin{cases} \mathfrak{U},\ \lambda \neq -n^2; \\ \{u \in \mathfrak{U} : \int\limits_{-\pi}^{\pi} u(x)\sin lx dx = 0,\ \lambda = l^2\}. \end{cases}$$

Theorem 10. [33] *For all $\lambda,\ \nu \in \mathbb{R}\setminus\{0\}$, the phase space of equation* (9) *is the union of two simple Banach C^∞-manifolds modeled by the space $\mathfrak{U}^1$.*

Therefore, it is shown that the phase space of equation (9) is the union of two connected components. In what follows, we denote by $\mathfrak{M}$ the component of this set that contains the zero point.

Theorem 11. *For any $\lambda \in \mathbb{R}\setminus\{0\},\ \nu \in \mathbb{R}_+$, in a neighborhood of the zero point, equation* (9) *has a finite-dimensional unstable invariant manifold $\mathfrak{M}^s$ and an infinite-dimensional stable invariant manifold $\mathfrak{M}^u$ modeled by the spaces $\mathfrak{U}^s$ and $\mathfrak{U}^u$, respectively.*

Conclusion

Numerical experiments on the solvability of linear stochastic Sobolev type equations are discussed in [19–21], and on stability — in [9–13]. We intend to carry out similar studies on the stability of semilinear stochastic equations.

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ИНВАРИАНТНЫЕ МНОГООБРАЗИЯ ПОЛУЛИНЕЙНЫХ УРАВНЕНИЙ СОБОЛЕВСКОГО ТИПА

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Статья посвящена обзору результатов автора по исследованию устойчивости полулинейных уравнений соболевского типа с относительно ограниченным оператором. Рассмотрены начально-краевые задачи для уравнений Хоффа, Осколкова нелинейной фильтрации жидкости, Осколкова плоскопараллельного течения жидкости, Бенжамина – Бона – Махони. Эти задачи при подходящем выборе функциональных пространств могут быть рассмотрены как частные случаи задачи Коши для полулинейного уравнения соболевского типа. При исследовании устойчивости мы пользуемся методами фазового пространства, основанными на теории вырожденных (полу)групп операторов, и применяем обобщение классической теоремы Адамара – Перрона. Показано существование устойчивого и неустойчивого инвариантных многообразий, моделируемых устойчивым и неустойчивым инвариантными пространствами линейной части уравнения, в случае, когда фазовое пространство является простым и относительный спектр и мнимая ось не имеют общих точек.

Ключевые слова: уравнения соболевского типа; инвариантные многообразия; уравнения Осколкова; уравнение Хоффа; уравнение Бенджамина – Бона – Махони.

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