MSC 35J20, 35J40 DOI: 10.14529/mmp220402

DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE WITH VARIABLE OPERATORS AND HOMOGENEOUS ROBIN BOUNDARY VALUE CONDITION IN UMD SPACES

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In this article, we give new results on the study of elliptic abstract second order differential equation with variable operators coefficients under the general Robin homogeneous boundary value conditions, in the framework of UMD spaces. Here, we do not assume the differentiability of the resolvent operators. However, we suppose that the family of variable operators verifies the Labbas–Terreni assumption inspired by the sum theory and similar to the Acquistapace–Terreni one. We use Dunford calculus, interpolation spaces and semigroup theory in order to obtain existence, uniqueness and maximal regularity results for the classical solution to the problem.

Keywords: Robin boundary value conditions; analytic semigroup; maximal regularity; Dunford operational calculus.

Introduction and Hypotheses

In a complex Banach space E, we consider the second-order differential equation with variable operator coefficients

$$u''(x) + A(x)\,u(x) - \omega u(x) = f(x), \qquad x \in (0,1)\,, \tag{1}$$

under the general Robin homogeneous boundary value conditions

$$\begin{cases} u'(0) - Hu(0) = 0, \\ u(1) = 0, \end{cases} \tag{2}$$

where $f \in L^p(0,1;E)$, $1 < p < +\infty$ and $(A(x))_{x\in[0,1]}$ is a family of closed linear operators in E with domains $D(A(x))$ that are dense in E. Here H is a closed linear operator in E, ω is a positive real number. The results proved here in the L^p case complete our recent paper concerning the hölderian case, see [1].

For all $x \in [0,1]$, set:

$$A_\omega(x) = A(x) - \omega I.$$

We seek for a classical solution to problem (1), (2), that is a function u such that:

$$\begin{cases} \text{for a.e } x \in (0,1), \quad u(x) \in D(A(x)) \text{ and} \\ x \mapsto A(x)\,u(x) \in L^p(0,1;E)\,, \\ u \in W^{2,p}(0,1;E)\,, \\ u(0) \in D(H)\,, \\ u \text{ satisfies (1), (2).} \end{cases} \tag{3}$$

The method is essentially based on Dunford's operational calculus, interpolation spaces, the semigroup theory and some techniques as in [1, 2].

Here we assume that

$$E \ \text{ is an } \ UMD \ \text{ space.} \tag{4}$$

We consider problem (1), (2) in an elliptic situation: the family of linear closed operators $(A(x))_{x\in[0,1]}$ satisfies

$\exists \omega_0 > 0$, $\exists C > 0 : \forall x \in [0,1]$, $\forall z \geqslant 0$, $(A_{\omega_0}(x) - zI)^{-1} \in \mathcal{L}(E)$ and:

$$\left\|(A_{\omega_0}(x) - zI)^{-1}\right\|_{\mathcal{L}(E)} \leqslant \frac{C}{1+z}. \tag{5}$$

It is well known that assumption (5) implies that for every $x \geqslant 0$ and every $\omega \geqslant \omega_0$, the square roots

$$Q_\omega(x) = -\left(-A_\omega(x)\right)^{1/2}$$

are well defined and generate analytic semigroups $\left(e^{yQ_\omega(x)}\right)_{y>0}$ that are not necessarily strongly continuous at 0 (see Balakrishnan [3] for dense domains and Martinez–Sanz [4] for non dense domains). We also assume:
$\exists C \geq 1,\ \theta_0 \in]0,\pi[$: $\forall s \in \mathbb{R}$, $\forall x \in [0,1]$, $\forall \omega \geqslant \omega_0$, $\left(-A_\omega(x)\right)^{is} \in L(E)$ and

$$\left\| \left(-A_\omega(x)\right)^{is} \right\|_E \leqslant C e^{\theta_0 |s|}, \tag{6}$$

$\exists C,\ \alpha,\ \mu > 0$: $\forall x,\ \tau \in [0,1]$, $\forall z \geqslant 0$, $\forall \omega \geqslant \omega_0$:

$$\begin{cases} \left\| A_\omega(x)\left(A_\omega(x) - zI\right)^{-1}\left(A_\omega(x)^{-1} - A_\omega(\tau)^{-1}\right)\right\|_{L(E)} \leqslant \dfrac{C\,|x-\tau|^\alpha}{|z+\omega|^\mu}, \\ \text{with } \alpha + 2\mu - 2 > 0. \end{cases} \tag{7}$$

The operators $Q(x)$ and H satisfy

$$D\left(Q(x)\right) \subset D(H)\ \forall x \in [0,1], \tag{8}$$

and the following commutativity condition:

$$\left(Q_\omega(x)\right)^{-1}\left(Q_\omega(x) - H\right)^{-1} = \left(Q_\omega(x) - H\right)^{-1}\left(Q_\omega(x)\right)^{-1} \forall x \in [0,1],\ \forall \omega \geq \omega_0. \tag{9}$$

Remark 1. For the complex power of an operator in (6), see for instance Haase [5]. Hypothesis (7) is well known as Labbas–Terreni assumption, see [6].

Remark 2. From (5) we deduce that, there exists $\theta_0 \in \left]0, \dfrac{\pi}{2}\right[$ and $r_0 > 0$ such that for all x belonging to $[0,1]$, the resolvent of $\left(A_{\omega_0}(x)\right)$ verifies:

$$\rho\left(A_{\omega_0}(x)\right) \supset \Pi_{\theta_0,r_0} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \theta_0\} \cup \overline{B(0,r_0)},$$

where $\overline{B(0,r_0)}$ is the closed ball of radius r_0 and centered in 0. We denote by Γ the boundary of Π_{θ_0,r_0} oriented from $\infty e^{i\theta_0}$ to $\infty e^{-i\theta_0}$.

The constant case ($A(x) = A$) with a Robin condition was treated by M. Cheggag *et al.* [7–9], for L^p and höelderian cases.

The commutator hypothesis (7) was used for the first time in Labbas [2] for the same problem but with boundary value conditions of Dirichlet type, in Bouziani *et al* [10] for transmission conditions and Haoua *et al* [1] for Robin conditions. All these studies were performed in the framework of hölderian spaces.

Another approach including the differentiability of the resolvents of $\left(A(x)\right)_{x\in[0,1]}$ was used by Da Prato and Grisvard [11], Labbas [2] and Boutaous *et al* [12]. Also, in these studies, the boundary value conditions considered were of Dirichlet type.

In this work, we consider an abstract homogeneous Robin boundary value condition in problem (1), (2) in L^p case.

Our main result is summarized by Theorem 1 in Subsection 4.2.

The paper is organized as follows. Section 2 contains some technical lemmas useful for the study of problem (1), (2). In Section 3, an heuristic reasoning is used to obtain a representation of the solution. We obtain an integral equation which is solved using (7). Section 4 is devoted to the study of the maximal regularity of the solution. In Section 5, the existence of the solution is proved using the associated approximating problem. Finally in section 6, we give a concrete example of partial differential equation to which our theory applies.

1. Technical Lemmas

Lemma 1. *There exists $C > 0$ such that for each $z \in \Gamma$ and $r > 0$, we have*

$$|z + r| \geqslant C\,|z|, \qquad |z + r| \geqslant Cr,$$
$$|z - r| \geqslant C\,|z|, \qquad |z - r| \geqslant Cr,$$

and

$$\forall r > 0,\ \forall \nu \in [0,1],\ \int_\Gamma \frac{|dz|}{|z \pm r|\,|z|^\nu} \leqslant \frac{C}{r^\nu}.$$

Proof. See [6, Lemma 6.1 and 6.2, p. 564]. □

Lemma 2. *Assume that (4) – (9) hold. Then there exist constants $C > 0$ and $\omega_1^* > \omega_0$ such that, for all $\omega \geqslant \omega_1^*$ and $x \in [0,1]$, the operator $Q_\omega(x) \pm H$ is boundedly invertible and*

$$\left\|(Q_\omega(x) \pm H)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{\sqrt{\omega}}.$$

Proof. See [13, Proposition 7, p. 987]. □

For al $\omega \geqslant \omega_1^*$ and $x \in [0,1]$, we consider the linear operator $\Pi_\omega(x)$ defined by

$$\Pi_\omega(x) = I + 2\left(I - e^{2Q_\omega(x)}\right)^{-1} Q_\omega(x)\, e^{2Q_\omega(x)} \left(Q_\omega(x) - H\right)^{-1}.$$

Proposition 1. *Under assumptions (4) – (6) and (9), for all $\omega \geqslant \omega_1^*$ and $x \in [0,1]$, $\Pi_\omega(x)$ is boundedly invertible.*

Proof. Here, our assumptions lead us to application of Lemma 5.2„ in [9, p. 1466] to obtain the desired result. □

For all $\omega \geqslant \omega_1^*$ and $x \in [0,1]$, we also define the linear operator $\Lambda_\omega(x)$ by

$$\begin{cases} D(\Lambda_\omega(x)) = D(Q_\omega(x)), \\ \Lambda_\omega(x) = Q_\omega(x) - H + e^{2Q_\omega(x)}(Q_\omega(x) + H),\ x \in [0,1], \end{cases}$$

which is the determinant of the system of our problem.

Lemma 3. *Assume (4) $\sim$ (6) and (9). Then for all $\omega \geqslant \omega_1^*$ and $x \in [0,1]$, $\Lambda_\omega(x)$ is closed and boundedly invertible with*

$$[\Lambda_\omega(x)]^{-1} = (Q_\omega(x) - H)^{-1}(\Pi_\omega(x))^{-1}\left(I - e^{2Q_\omega(x)}\right)^{-1}.$$

Proof. See Lemma 2.5 in [1, p. 4]. □

Lemma 4. *From (5) and (7), we have*

$$\begin{cases} \exists C, \alpha, \mu > 0 : \forall x, \tau \in [0,1], \forall z \geqslant 0, \forall \omega \geqslant \omega_1^*, \\ \left\|Q_\omega(x)(Q_\omega(x) - zI)^{-1}\left(Q_\omega(x)^{-1} - Q_\omega(\tau)^{-1}\right)\right\|_{L(E)} \leqslant \dfrac{C\,|x - \tau|^\alpha}{|z + \omega|^\mu}, \\ \text{with } \alpha + 2\mu - 2 > 0. \end{cases}$$

Proof. For all $z \geqslant 0$, $x \in [0,1]$ and $\omega \geqslant \omega_1^*$, we have

$$\left(-\sqrt{-A_\omega(x)} - zI\right)^{-1} = \frac{-1}{2\pi i}\int_\Gamma \frac{(A_\omega(x) - z''I)^{-1}}{\sqrt{-z''} + z}dz'', \tag{10}$$

and using the simple calculus we obtain

$$- \left(-A_\omega(x)\right)^{-1/2} + \left(-A_\omega(\tau)\right)^{-1/2} =$$
$$= \frac{-1}{2\pi i} \int\limits_{\Gamma} \frac{A_\omega(x)\left(A_\omega(x) - z'I\right)^{-1}\left[A_\omega(x)^{-1} - A_\omega(\tau)^{-1}\right] A_\omega(\tau)\left(A_\omega(\tau) - z'I\right)^{-1}}{\sqrt{-z'}} dz'.$$

Here we use formula (10) for $\left(-\sqrt{-A_\omega(x)} - zI\right)^{-1}$ with a curve Γ' homotopic to Γ and $\sqrt{-A_\omega(x)}$ is written as $\sqrt{-z''}$ by Dunford's calculus, we obtain

$$Q_\omega(x)\left(Q_\omega(x) - zI\right)^{-1}\left(Q_\omega(x)^{-1} - Q_\omega(\tau)^{-1}\right) = \frac{-1}{2\pi i} \int\limits_{\Gamma'} \int\limits_{\Gamma} \frac{\sqrt{-z''} A_\omega(x)\left(A_\omega(x) - z''I\right)^{-1}}{\sqrt{-z'}\left(\sqrt{-z''} + z\right)} \times$$
$$\times \left(A_\omega(x) - z'I\right)^{-1}\left[A_\omega(x)^{-1} - A_\omega(\tau)^{-1}\right] A_\omega(\tau)\left(A_\omega(\tau) - z'I\right)^{-1} dz'dz''.$$

Using the algebraic identity

$$A_\omega(x)\left(A_\omega(x) - z''I\right)^{-1}\left(A_\omega(x) - z'I\right)^{-1} = \frac{A_\omega(x)\left(A_\omega(x) - z''I\right)^{-1}}{z'' - z'} - \frac{A_\omega(x)\left(A_\omega(x) - z'I\right)^{-1}}{z'' - z'},$$

then, we obtain

$$Q_\omega(x)\left(Q_\omega(x) - zI\right)^{-1}\left(Q_\omega(x)^{-1} - Q_\omega(\tau)^{-1}\right) = \tfrac{1}{2\pi i} \int\limits_{\Gamma} \tfrac{1}{\sqrt{-z'}} \times$$
$$\times \left[\int\limits_{\Gamma'} \tfrac{\sqrt{-z''} A_\omega(x)(A_\omega(x) - z''I)^{-1}\left[A_\omega(x)^{-1} - A_\omega(\tau)^{-1}\right] A_\omega(\tau)(A_\omega(\tau) - z'I)^{-1}}{\left(\sqrt{-z''} + z\right)(z'' - z')} dz''\right] dz' +$$
$$+ \tfrac{1}{2\pi i} \int\limits_{\Gamma'} \left[\int\limits_{\Gamma} \tfrac{A_\omega(x)(A_\omega(x) - z'I)^{-1}\left[A_\omega(x)^{-1} - A_\omega(\tau)^{-1}\right] A_\omega(\tau)(A_\omega(\tau) - z'I)^{-1}}{\sqrt{-z'}(z'' - z')} dz'\right] \tfrac{\sqrt{-z''} dz''}{\sqrt{-z''} + z} = I_1 + I_2,$$

we have $I_2 = 0$ and

$$\frac{1}{\sqrt{-z''} + z} = \frac{1}{\sqrt{-z'} + z} \quad \text{when } z'' \to z',$$

then

$$\|I_1\|_E \leqslant C \int\limits_{\Gamma} \tfrac{|x - \tau|^\alpha d|z'|}{|z'|^\mu\left(|z'|^{1/2} + z\right)}.$$

Setting $z' = z\xi$, we obtain

$$\|I_1\|_E \leqslant \tfrac{C|x - \tau|^\alpha}{|z|^\mu} \int\limits_{\Gamma} \tfrac{d|\xi|}{|\xi|^{1 + (\alpha/2 + \mu - 1)}} \leq \tfrac{C|x - \tau|^\alpha}{|z|^\mu}.$$

□

2. Representation of Solution

Let us recall briefly the case when $A_\omega(x) = A - \omega I$ is a constant operator satisfying the natural ellipticity hypothesis mentioned above (we take $Q_\omega = -\left(-A_\omega\right)^{1/2}$). In this case, the representation of the solution v to problem (1), (2) is given by the formula (see [9])

$$v(x)=\frac{1}{2}e^{xQ_\omega}(Q_\omega+H)(\Lambda_\omega)^{-1}Q_\omega^{-1}\int_0^1 e^{sQ_\omega}f(s)\,ds-\frac{1}{2}e^{xQ_\omega}(Q_\omega+H)(\Lambda_\omega)^{-1}e^{Q_\omega}Q_\omega^{-1}\times$$
$$\times\int_0^1 e^{(1-s)Q_\omega}f(s)\,ds-\tfrac{1}{2}e^{(1-x)Q_\omega}(Q_\omega+H)(\Lambda_\omega)^{-1}e^{Q_\omega}Q_\omega^{-1}\int_0^1 e^{sQ_\omega}f(s)-$$
$$-\tfrac{1}{2}e^{(1-x)Q_\omega}\left[I-(Q_\omega+H)(\Lambda_\omega)^{-1}e^{2Q_\omega}\right]Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega}f(s)\,ds+$$
$$+\tfrac{1}{2}Q_\omega^{-1}\int_0^x e^{(x-s)Q_\omega}f(s)\,ds+\tfrac{1}{2}Q_\omega^{-1}\int_x^1 e^{(s-x)Q_\omega}f(s)\,ds.$$

Set

$$L_{Q_\omega(x)}(x,f)=\tfrac{1}{2}e^{xQ_\omega(x)}(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}Q_\omega(x)^{-1}\int_0^1 e^{sQ_\omega(x)}f(s)\,ds-$$
$$-\tfrac{1}{2}e^{xQ_\omega(x)}(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}e^{Q_\omega(x)}Q_\omega^{-1}(x)\int_0^1 e^{(1-s)Q_\omega(x)}f(s)-$$
$$-\tfrac{1}{2}e^{(1-x)Q_\omega(x)}(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}e^{Q_\omega(x)}Q_\omega(x)^{-1}\int_0^1 e^{sQ_\omega(x)}f(s)\,ds-$$
$$-\tfrac{1}{2}e^{(1-x)Q_\omega(x)}\left[I-(Q_\omega(x)+H)(\Lambda_\omega(x))^{-1}e^{2Q_\omega(x)}\right]Q_\omega^{-1}(x)\int_0^1 e^{(1-s)Q_\omega(x)}f(s)$$
$$+\tfrac{1}{2}Q_\omega(x)^{-1}\int_0^x e^{(x-s)Q_\omega(x)}f(s)\,ds+\tfrac{1}{2}Q_\omega(x)^{-1}\int_x^1 e^{(s-x)Q_\omega(x)}f(s)\,ds.$$

We can write that:

$$L_{Q_\omega(x)}(x,f)=L_{Q_\omega(x)}(x,u''(x)+A_\omega(x)u(x)).$$

After two integrations by parts and some formal calculus, as in [1], we obtain the following abstract equation:

$$w+P_\omega w=G(x,f),$$

where

$$w(\cdot)=A_\omega(\cdot)u(\cdot).$$

Here, for all $x\in[0,1]$, $\omega\geqslant\omega_1^*$

$$(P_\omega w)(x)=\tfrac{1}{2}K_\omega(x)e^{xQ_\omega(x)}\int_0^1 Q_\omega(x)^3e^{sQ_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds-$$
$$-\tfrac{1}{2}e^{xQ_\omega(x)}K_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{(1-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds-$$
$$-\tfrac{1}{2}e^{(1-x)Q_\omega(x)}K_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{sQ_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds+$$
$$+\tfrac{1}{2}e^{(1-x)Q_\omega(x)}K_\omega(x)e^{2Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{(1-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds-$$
$$-\tfrac{1}{2}e^{(1-x)Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{(1-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds+$$
$$+\tfrac{1}{2}\int_0^x Q_\omega(x)^3e^{(x-s)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds+$$
$$+\tfrac{1}{2}\int_x^1 Q_\omega(x)^3e^{(s-x)Q_\omega(x)}\left(Q_\omega(s)^{-2}-Q_\omega(x)^{-2}\right)w(s)\,ds=\sum_{i=1}^{7}I_i(x),$$

where

$$K_\omega(x) = (Q_\omega(x) + H)[\Lambda_\omega(x)]^{-1},$$

and

$$G_{Q_\omega(x)}(x, f)(x) = -A_\omega(x) L_{Q_\omega(x)}(x, f).$$

Proposition 2. *Assume (4) – (9). Then for all $\omega \geqslant \omega_1^*$:*

$$\|P_\omega\|_{\mathcal{L}(L^p(0,1;E))} \leqslant \frac{1}{2}.$$

Proof. Set for all $(x - s) > 0$ and z outside the negative axis:

$$\begin{cases} \varphi(z) = z^3 e^{z(x-s)} \\ \psi(z) = -\sqrt{z}. \end{cases}$$

The functional calculation applies to the sector operator $-A_\omega(x)$ for each $x \in [0, 1]$ and each holomorphic function g satisfying the right hypotheses, see [5], p. 30, so we have

$$g(-A_\omega(x)) = \frac{1}{2i\pi} \int_{\Gamma'} g(z)(-A_\omega(x) - zI)^{-1} dz. \tag{11}$$

Here Γ' is the opposite of Γ. We apply Haase's rule (see [5]) of composition for each $x \in [0, 1]$, we have

$$[\varphi \circ \psi](-A_\omega(x)) = \varphi[\psi(-A_\omega(x))] = \varphi\left[-\sqrt{-A_\omega(x)}\right] = Q_\omega(x)^3 e^{(x-s)Q_\omega(x)},$$

and using (11), we have

$$[\varphi \circ \psi](-A_\omega(x)) = (-A_\omega(x)) = \frac{1}{2i\pi} \int_{\Gamma'} [\varphi \circ \psi](z)(-A_\omega(x) - zI)^{-1} dz,$$

therefore

$$Q_\omega(x)^3 e^{(x-s)Q_\omega(x)} = \tfrac{1}{2i\pi} \int_{\Gamma'} (-\sqrt{z})^3 e^{-\sqrt{z}(x-s)} (-A_\omega(x) - zI)^{-1} dz,$$

and in fact the change of variable $z \mapsto -z$ to get back to our curve:

$$Q_\omega(x)^3 e^{(x-s)Q_\omega(x)} = \tfrac{1}{2i\pi} \int_{\Gamma} \left(\sqrt{-z}\right)^3 e^{-\sqrt{-z}(x-s)} (A_\omega(x) - zI)^{-1} dz.$$

So

$$\begin{gathered} Q_\omega(x)^3 e^{(x-s)Q_\omega(x)} \left(Q_\omega(s)^{-2} - Q_\omega(x)^{-2}\right) \\ = \tfrac{1}{2i\pi} \int_{\Gamma} \sqrt{-z} e^{-\sqrt{-z}(x-s)} A_\omega(x)(A_\omega(x) - zI)^{-1} \left(A_\omega(s)^{-1} - A_\omega(x)^{-1}\right) dz. \end{gathered}$$

Consider for example $I_6(x) + I_7(x)$ in P_ω, we can write for all $x \in [0, 1]$ and $\omega \geqslant \omega_1^*$:

$$I_6(x) + I_7(x) = \mathcal{L}_\omega w(x) = \tfrac{1}{4i\pi} \int_{\Gamma} \int_0^1 K^1_{\sqrt{-z}}(x, s) w(s) ds dz,$$

and

$$L_\omega w(x) = \int_0^1 K^1_{\sqrt{-z}}(x, s) w(s) ds,$$

where

$$K^1_{\sqrt{-z}}(x, s) = \begin{cases} \sqrt{-z} e^{-\sqrt{-z}(x-s)} A_\omega(x)(A_\omega(x) - zI)^{-1} \left(A_\omega(s)^{-1} - A_\omega(x)^{-1}\right) & \text{in } s \leqslant x \\ \sqrt{-z} e^{-\sqrt{-z}(s-x)} A_\omega(x)(A_\omega(x) - zI)^{-1} \left(A_\omega(s)^{-1} - A_\omega(x)^{-1}\right) & \text{in } s \geqslant x. \end{cases}$$

We should estimate

$$\left\| L_\omega \right\|_{L(L^p(0,1;E)}.$$

In virtue of the Schur's lemma, we write

$$\int_0^1 \left\| K^1_{\sqrt{-z}}(x,s) \right\| ds \leqslant C \frac{|z|^{1/2}}{|z+\omega|^\mu} \left(\int_0^1 e^{-Re\sqrt{-z}|x-s|} |x-s|^\alpha ds \right).$$

Due to Hölder's inequality, we have

$$\int_0^x e^{-Re\sqrt{-z}(x-s)} (x-s)^\alpha ds + \int_x^1 e^{-Re\sqrt{-z}(s-x)} (s-x)^\alpha ds \leqslant= \left(\frac{1}{Re\sqrt{-z}}\right)^{1+\alpha};$$

where we used

$$\int_0^x e^{-Re\sqrt{-z}(x-s)} (x-s)\, ds \leqslant \frac{1}{\left(Re\sqrt{-z}\right)^2},$$

and

$$\int_x^1 e^{-Re\sqrt{-z}(s-x)} (s-x)\, ds \leqslant \frac{1}{\left(Re\sqrt{-z}\right)^2}.$$

Not that the function $u \longmapsto 1 - ue^{-u} - e^{-u}$ is greater than or equal to 1 for $u \geqslant 0$. From where

$$\begin{aligned} \int_0^1 \left\| K_{\sqrt{-z}}(x,s) \right\| ds &\leqslant C \tfrac{|z|^{1/2}}{|z+\omega|^\mu} \left(\int_0^1 e^{-Re\sqrt{-z}|x-s|} |x-s|^\alpha ds \right) \\ &\leqslant C \tfrac{|z|^{1/2}}{|z+\omega|^\mu} \tfrac{1}{|z|^{1/2+\alpha/2}} \leqslant \tfrac{C}{|z||z+\omega|^{\mu+\alpha/2-1}}, \end{aligned}$$

and so

$$\left\| L_\omega \right\|_{L(L^p(0,1;E)} \leqslant \frac{C}{|z|\,|z+\omega|^{\mu+\alpha/2-1}},$$

Finally we obtain

$$\left\| \mathcal{L}_\omega \right\|_{L(L^p(0,1;E)} \leqslant \int_\gamma \frac{C}{|z|\,|z+\omega|^{\mu+\alpha/2-1}} |dz| = O\left(\frac{1}{|\omega|^{\mu+\alpha/2-1}} \right).$$

The same technique is used for the other terms.

Therefore for all $\omega \geqslant \omega_1^*$, $\|P_\omega\|_{\mathcal{L}(L^p(0,1;E))} \leqslant \dfrac{1}{2}$ which leads us to invert $I + P_\omega$ in the space $L^p(0,1;E)$. We can write for all $\omega \geqslant \omega_1^*$ and $x \in [0,1]$

$$u(x) = A_\omega(x)^{-1} (I + P_\omega)^{-1} G_{Q_\omega(x)}(x,f). \tag{12}$$

□

3. Regularity of Solution

Throughout this section we assume that $\omega \geqslant \omega_1^*$.

3.1. Regularity of Second Member $G_{Q_\omega(x)}(x,f)$

As in the autonomous case we use the following lemma, see [14].

Lemma 5. *Fix $x \in [0,1]$, $p \in]1,\infty[$ and $\omega \geqslant \omega_1^*$. Then*
1. $t \longmapsto A_\omega(x) e^{tQ_\omega(x)}\varphi \in L^p(0,1;E)$ if and only if $\varphi \in (D(A(x)),E)_{\frac{1}{2p},p}$.
2. $t \longmapsto Q_\omega(x) e^{tQ_\omega(x)}\varphi \in L^p(0,1;E)$ if and only if $\varphi \in (D(A(x)),E)_{\frac{1}{2p}+\frac{1}{2},p}$.

We obtain then the following regularity results of $G_{Q_\omega(x)}(x,f)$.

Proposition 3. *Assume (4) – (9) and $f \in L^p(0,1;E)$ with $1 < p < \infty$. Then for all $\omega \geqslant \omega_1^*$*

$$G_{Q_\omega(x)}(x,f) \in L^p(0,1;E).$$

Proof. Let $x \in [0,1]$ and $\omega \geqslant \omega_1^*$, we have

$$G_{Q_\omega(x)}(x,f) = -\tfrac{1}{2} Q_\omega(x) e^{xQ_\omega(x)} K_\omega(x) \int_0^1 e^{sQ_\omega(x)} f(s)\,ds +$$
$$+\tfrac{1}{2} Q_\omega(x) e^{(1-x)Q_\omega(x)} \int_0^1 e^{(1-s)Q_\omega(x)} f(s)\,ds - \tfrac{1}{2} Q_\omega(x) \int_0^x e^{(x-s)Q_\omega(x)} f(s)\,ds -$$
$$-\tfrac{1}{2} Q_\omega(x) \int_x^1 e^{(s-x)Q_\omega(x)} f(s)\,ds + R(x,f) = \sum_{i=1}^4 J_i(x) + R(x,f),$$

where

$$R(x,f) = \tfrac{1}{2} Q_\omega(x) e^{xQ_\omega(x)} K_\omega(x) \int_0^1 e^{(2-s)Q_\omega(x)} f(s)\,ds + \tfrac{1}{2} Q_\omega(x) e^{(1-x)Q_\omega(x)} K_\omega(x) \times$$
$$\times \int_0^1 e^{(1+s)Q_\omega(x)} f(s)\,ds - \tfrac{1}{2} Q_\omega(x) e^{(1-x)Q_\omega(x)} K_\omega(x) \int_0^1 e^{(3-s)Q_\omega(x)} f(s)\,ds.$$

For any $\zeta \in E$, $k \in \mathbb{N}$, we have $e^{Q_\omega(x)}\zeta \in D\left(\left(Q_\omega(x)^k\right)\right)$, so

$$A_\omega(x) e^{\cdot Q_\omega(x)} e^{Q_\omega(x)}\zeta = e^{\cdot Q_\omega(x)} A_\omega(x) e^{Q_\omega(x)}\zeta,$$

and $s \longmapsto A_\omega(x) e^{sQ_\omega(x)} e^{Q_\omega(x)}\zeta$ is bounded and thus in $L^p(0,1;E)$. To conclude it is enough to remark that $R(.,d_0,u_1,f)$ can be written as a sum of terms $PA_\omega(x) e^{\cdot Q_\omega(x)} e^{Q_\omega(x)}\zeta$, $PA_\omega(x) e^{(1-\cdot)Q_\omega(x)} e^{Q_\omega(x)}\zeta$, where $P \in L(E)$, $\zeta \in E$, see Lemma 9 in [13].
For J_3, we consider the following problem:

$$\begin{cases} \psi'(x) - Q_\omega(x)\psi(x) = f(x), & x \in (0,1), \\ \psi(0) = 0. \end{cases} \tag{13}$$

Let ψ be the strict solution to problem (13). Fix $x \in [0,1]$, and set

$$v(s) = e^{(x-s)Q_\omega(x)}\psi(s), \quad s \in [0,x].$$

Then for each $s \in [0,x]$, we have

$$v'(s) = -Q_\omega(x) e^{(x-s)Q_\omega(x)}\psi(s) + e^{(x-s)Q_\omega(x)}\left[Q_\omega(s)\psi(s) + f(s)\right] =$$
$$= Q_\omega(x) e^{(x-s)Q_\omega(x)}\left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right] Q_\omega(s)\psi(s) + e^{(x-s)Q_\omega(x)} f(s).$$

Integrate over $]0,x[$ and apply $Q_\omega(x)$ to both sides, the result is

$$Q_\omega(x)\psi(x) = \int_0^x Q_\omega(x)^2 e^{(x-s)Q_\omega(x)}\left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right] Q_\omega(s)\psi(s)\,ds + Q_\omega(x) \times$$
$$\times \int_0^x e^{(x-s)Q_\omega(x)} f(s)\,ds = \int_0^x Q_\omega(x)^2 e^{(x-s)Q_\omega(x)}\left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right] Q_\omega(s)\psi(s)\,ds + J_3(x);$$

see Acquistapace and Terreni [15, p. 56, 57]. Due to Monniaux Theorem 5.11 in [16, p. 59], we have $x \longmapsto Q_\omega(x)\psi(x)$ in $L^p(0,1;E)$ and due to Lemma 4, we have $x \longmapsto$

$\int_0^x Q_\omega(x)^2 e^{(x-s)Q_\omega(x)} \left[Q_\omega(x)^{-1} - Q_\omega(s)^{-1}\right] Q_\omega(s)\,\psi(s)\,ds$ in $L^p(0,1;E)$. Then $x \longmapsto J_3(x)$ is in $L^p(0,1;E)$.
The same technique is used for the other terms. Finally

$$x \longmapsto G_{Q_\omega(x)}(x,f) \in L^p(0,1;E).$$

□

3.2. Regularity of P_ω

Proposition 4. *Assume (4) – (9). Then for all $\omega \geqslant \omega_1^*$, we have*

$$P_\omega \in \mathcal{L}\left(L^p(0,1;E), L^p(0,1;E) \cap B\left(0,1;D_{A(\cdot)}\left(\frac{\beta}{2},+\infty\right)\right)\right), \text{ where } \beta \in \left]0, \alpha+2\mu-2\right].$$

Proof. Let us prove that

$$\sup_{r>0} \left\| r^\beta A_\omega(x)\left(A_\omega(x)-rI\right)^{-1}\left(P_\omega w\right)(x)\right\|_E \leqslant C\left\|w\right\|_{L^p(0,1;E)}.$$

Using the following identity

$$\begin{gathered}A_\omega(x)\left(A_\omega(x)-rI\right)^{-1} A_\omega(x)\left(A_\omega(x)-zI\right)^{-1} = \\ = \tfrac{-z}{r-z} A_\omega(x)\left(A_\omega(x)-zI\right)^{-1} + \tfrac{r}{r-z} A_\omega(x)\left(A_\omega(x)-rI\right)^{-1},\end{gathered}$$

we have

$$\begin{gathered}r^\beta A_\omega(x)(A_\omega(x)-rI)^{-1}(P_\omega w)(x) = \tfrac{r^\beta}{4\pi i} K_\omega(x) \int_\Gamma \int_0^1 e^{-\sqrt{-z}(x+s)} \tfrac{z\sqrt{-z}}{r-z} A_\omega(x) \times \\ \times (A_\omega(x)-zI)^{-1}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) w(s)\,ds dz - \tfrac{r^\beta}{4\pi i} \int_\Gamma \int_0^1 e^{-\sqrt{-z}(2-x-s)} \tfrac{z\sqrt{-z}}{r-z} A_\omega(x) \times \\ \times (A_\omega(x)-zI)^{-1}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) w(s)\,ds dz + \tfrac{r^\beta}{4\pi i} \int_\Gamma \int_0^x e^{-\sqrt{-z}(x-s)} \tfrac{z\sqrt{-z}}{r-z} A_\omega(x) \times \\ \times (A_\omega(x)-zI)^{-1}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) w(s)\,ds dz + \tfrac{r^\beta}{4\pi i} \int_\Gamma \int_x^1 e^{-\sqrt{-z}(s-x)} \tfrac{z\sqrt{-z}}{r-z} A_\omega(x) \times \\ \times (A_\omega(x)-zI)^{-1}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) w(s)\,ds dz - \tfrac{r^\beta}{2} A_\omega(x)^2 (A_\omega(x)-rI)^{-1} e^{xQ_\omega(x)} \times \\ \times K_\omega(x)\, e^{Q_\omega(x)} \int_0^1 e^{(1-s)Q_\omega(x)}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) w(s)\,ds - \tfrac{r^\beta}{2} A_\omega(x)^2 (A_\omega(x)-rI)^{-1} \times \\ \times e^{(1-x)Q_\omega(x)} K_\omega(x)\, e^{Q_\omega(x)} \int_0^1 e^{sQ_\omega(x)}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) w(s)\,ds + \tfrac{r^\beta}{2} A_\omega(x)^2 (A_\omega(x)-rI)^{-1} \times \\ \times e^{(1-x)Q_\omega(x)} K_\omega(x)\, e^{2Q_\omega(x)} \int_0^1 e^{(1-s)Q_\omega(x)}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) w(s)\,ds = \sum_{i=1}^7 \Phi_i(x).\end{gathered}$$

For $\Phi_1(x)$, we can write:

$$\Phi_1(x) = \int_0^1 K_2(x,s)\,w(s)\,ds,$$

where

$$K_2(x,s) = \frac{r^\beta}{4\pi i} \int_\Gamma K_\omega(x)\, e^{-\sqrt{-z}(x+s)} \frac{z\sqrt{-z}}{r-z} A_\omega(x)\left(A_\omega(x)-zI\right)^{-1}\left(A_\omega(s)^{-1}-A_\omega(x)^{-1}\right) dz.$$

Then, we have

$$\int_0^1 |K_2(x,s)|\,ds \leqslant Cr^{\beta}\int_0^1\int_\Gamma \frac{|z|^{3/2}}{|r-z|}e^{-c_0|z|^{1/2}(x+s)}\frac{|x-s|^{\alpha}}{|z|^{\mu}}d\,|z|\,ds \leqslant$$
$$\leqslant Cr^{\beta}\int_\Gamma \frac{|z|^{3/2}}{|r-z|}\left(\sup_{x\in[0,1]}\int_0^x e^{-c_0|z|^{1/2}(x+s)}(x-s)^{\alpha}ds+\sup_{x\in[0,1]}\int_x^1 e^{-c_0|z|^{1/2}(x+s)}(s-x)^{\alpha}ds\right)\frac{d|z|}{|z|^{\mu}} \leqslant$$
$$\leqslant Cr^{\beta}\int_\Gamma \frac{|z|^{3/2}}{|r-z||z|^{\mu}}\frac{d|z|}{|z|^{1/2+\alpha/2}},$$

which leads us to using of Lemma 1, hence

$$\int_0^1 |K_2(x,s)| \leqslant Cr^{\beta}\int_\Gamma \frac{d\,|z|}{|r-z|\,|z|^{\alpha/2+\mu-1}} \leqslant \frac{Cr^{\beta}}{r^{\alpha/2+\mu-1}}.$$

In virtue of Schur's lemma, we conclude that

$$\|\Phi_1\|_{L^p(0,1;E)} \leqslant \frac{Cr^{\beta}}{r^{\alpha/2+\mu-1}}\|w\|_{L^p(0,1;E)}.$$

The same technique is used for the other terms $\Phi_i(x)$, $i = 2, 3, ..., 7$.

□

Summarizing the above results we obtain the following theorem.

Theorem 1. *Assume (4) – (9). Let* $f \in L^p(0,1;E)$ *with* $1 < p < +\infty$ *and for all* $\omega \geqslant \omega_1^*$. *Then equation (12) has a unique solution* $w(\cdot) = A_\omega(\cdot)\,u(\cdot)$ *such that*
1) $A_\omega(\cdot)\,u(\cdot) \in L^p(0,1;E)$;
2) $u'' \in W^{2,p}(0,1;E)$.

Proof. We have

$$\begin{aligned} u''(\cdot) &= f(\cdot) + A_\omega(\cdot)\,u(\cdot)\\ &= f(\cdot) + \left[G_{Q_\omega(x)}(d_0,u_1,f)(\cdot) - (P_\omega w)(\cdot)\right]\\ &= \left[f(\cdot) + G_{Q_\omega(x)}(d_0,u_1,f)(\cdot)\right] - (P_\omega w)(\cdot). \end{aligned}$$

□

4. Approximating Problem

In Section 2, we supposed the existence of an exact solution to problem (1), (2) and by using a heuristic reasoning we constructed a representation of the solution. Now to prove the existence of the solution, we consider the following approximating problem

$$\begin{cases} u_n''(x) + A_n(x)u_n(x) - \omega u_n(x) = f(x), & x \in]0,1[,\\ u_n'(0) - Hu_n(0) = 0, \\ u_n(0) = 0, \end{cases} \tag{14}$$

where $(A_n(x))_{x\in[0,1]}$ is the family of Yosida approximations of $(A(x))_{x\in[0,1]}$ defined by

$$A_n(x) = -nA(x)\,(A(x) - nI)^{-1},\ \ n \in \mathbb{N}^*.$$

We use the same arguments as in [1, 2, 10], to show that: $u_n \to u$.

5. Concrete General Example

Consider the complex Banach space $E = L^q(0,1)$, $1 < q < +\infty$ with its usual norm. Then E is an UMD (Unconditional Martingale Differences) Banach space. Define the family of closed linear operators $A(x)$ for all $x \in [0,1]$ by

$$\begin{cases} D(A(x)) = \{\varphi \in W^{2,q}((0,1)) : \varphi(0) = 0,\ a(x)\varphi(1) + b(x)\varphi'(1) = 0\}, \\ (A(x)\varphi)(y) = \varphi''(y), \quad y \in (0,1), \end{cases}$$

where $a(\cdot)$ and $b(\cdot)$ are two real functions in $C^1([0,1])$ such that:

$$a(x) \geqslant 0,\ b(x) \geqslant 0 \ \text{ and } \inf_{x\in[0,1]} (a(x) + b(x)) > 0.$$

Let us define the linear operator H by $H = \alpha I$, where $\alpha > 0$.
By direct calculations, one proves hypotheses (5) and (7) – (9), see Agmon, Douglis and Niremberg [17]. Hypotheses (6) is proved in Labbas–Moussaoui [18]. Then Theorem 1 applies to the following concrete quasi-elliptic boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) - \omega(x,y) = f(x,y), \quad (x,y) \in (0,1)\times(0,1), \\ \dfrac{\partial u}{\partial x}(0,y) - \alpha u(0,y) = 0, \quad u(1,y) = 0, \quad y \in (0,1), \\ a(x)\,u(x,1) + b(x)\dfrac{\partial u}{\partial y}(x,1) = 0, \quad u(x,0) = 0. \end{cases}$$

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Received January 7, 2022
Revised July 6, 2022

УДК 521.3 **DOI: 10.14529/mmp220402**

ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ ЭЛЛИПТИЧЕСКОГО ТИПА С ПЕРЕМЕННЫМИ ОПЕРАТОРАМИ И ОДНОРОДНЫМ ГРАНИЧНЫМ УСЛОВИЕМ РОБЕНА В ПРОСТРАНСТВАХ УМД

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В этой статье мы изучаем абстрактное дифференциальное уравнение эллиптического типа второго порядка с переменными операторными коэффициентами и общими граничными условиями Робина в основе UMD пространств. Результаты существования и регулярности получаются когда предположение Лаббаса – Террени выполняется с использованием теории полугрупп и интерполяционных пространств.

Ключевые слова: граничные условия Робина; аналитическая полугруппа; максимальная регулярность; Данфорд операционное исчисление.

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Поступила в редакцию 7 января 2022 г.
Принято к публикации 6 июля 2022 г.