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# RECOVERING OF THE HEAT TRANSFER COEFFICIENT FROM THE TEMPERATURE MEASUREMENTS

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An inverse analysis is used to recover the heat transfer coefficient in heat conduction problems from boundary measurement of the temperature. The numerical scheme is based on the finite element method in the space variables, the method of finite differences in time, and a special iteration scheme to determine the heat transfer coefficients on each time step. The heat transfer coefficients is sought in the form of a finite segment of a series with unknown Fourier coefficients depending on time. The algorithm for solving the problem relies on theoretical results stating that this problem is well-posed and can be reduced to an operator equation with a contraction. The results of numerical experiments confirm theoretical arguments that this problem is indeed well-posed. The obtained results reveal the accuracy, efficiency, and robustness of the proposed algorithm. It is stable under random perturbations of the data.

*Keywords: inverse problem; heat transfer coefficient; parabolic equation; heat and mass transfer.*

## Introduction

Consider an inverse problem of recovering the heat transfer coefficient. The mathematical model in question is as follows:

$$Mu = u_t - Lu = f(t,x), \quad (t,x) \in Q = (0,T) \times G, \tag{1}$$

$$Bu|_S = g(t,x) \ \ (S = (0,T) \times \Gamma), \quad u|_{t=0} = u_0(x), \tag{2}$$

$$u(t,b_i) = \psi_i(t) \ \ (i = 1,2,\dots,r), \tag{3}$$

where $Lu = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_ix_j} + \sum_{i=1}^{n} a_i(t,x)u_{x_i} + a_0(t,x)u$, $G \in \mathbb{R}^n$ is a bounded domain with the boundary $\Gamma$, $Bu = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_i}\nu_j + (\beta(t,x) + \beta_0(t,x))u$, ($\nu$ is the outward unit normal to $\Gamma$), and $b_i \in \Gamma$, $\{b_i\}_{i=1}^{r}$ is a collection of points. The problem is to find a solution to equation (1) satisfying (2), (3) and the unknown function $\beta(t,x) = \sum_{j=1}^{r} \beta_i(t)\Phi_i(t,x)$ (the functions $\Phi_i$ are given and $\beta_i(t)$ are unknowns).

In many thermal engineering problems involving high temperatures/ high pressures, ecology, diffusion, filtration, the boundary conditions are not fully known since there are technical difficulties in obtaining such data (see [1–4] and the bibliography therein). To recover the desired parameters, inverse problem formulations are required, which entail to performing some extra measurements of certain accessible and relevant quantities. At present, there are many works dealing with numerically solving problem (1) – (3), the points $\{b_i\}$ in (3) can either be interior [5–13] or boundary ones [14, 15] in $G$. Describe some known results. In the stationary case the above statement is considered in [16]. In [15], a parabolic system is considered and the constant heat transfer coefficients are sought (the uniqueness theorems for the solutions is obtained and the numerical method is described).

The time-dependent heat transfer coefficient is numerically determined from the values of the solution in a set of internal points in [7–9, 11–13]. In [5, 6], a system of two one-dimensional parabolic equations is considered. In [7,17], it is assumed that the heat transfer coefficient depends on $x$ and the additional data are the Dirichlet condition on a part of the lateral surface of a cylinder. The data of the same type are used in [18]. There are several studies, where the heat transfer coefficient is determined in the case of nonlinear boundary conditions. We can refer to the classical results [19], where the uniqueness theorem of a classical solution was obtained, and can also mention the articles [20–25]), where the nonlinear function $\varphi(t, x, u)$ (sometimes independent on the space variables) is sought under the boundary condition of the form $\frac{\partial u}{\partial N} + \varphi(t, x, u) = g$. Note that in many cases the problem is replaced with the corresponding control problem which is actually studied but a solution to the latter problem is not a solution to the initial problem, i. e., the convex set over which the minimization is performed is sufficiently small (see [23–25]). Many works also deal with the closely related problem of determining the fluxes through the boundary or a part of it using the same overdetermination conditions (3), which arises when linearizing the problem (see [4, 26–28] and the references therein). There are very few theoretical results for problem (1) – (3) especially in the multidimensional case. We can refer to the article [29], where the uniqueness theorems was obtained for classical solutions in the case of $Mu = u_t - \Delta u = f$ and $r = 1$. The existence theorem in the multidimensional case for solutions to problem (1) – (3) was obtained in [30].

Note that almost all numerical algorithm used in the above-mentioned article are based on reducing the problem to a control problem and minimization of the corresponding cost functional which often requires large computational capabilities. Some of the methods in model cases employ explicit representations of Green functions or Fourier expansions of solutions. Moreover, the heat transfer coefficient depends on $t$ only in almost all these articles. In our opinion, it is quite naturally to look for this coefficient in the form of a finite segment of a Fourier series or using the basis of the finite element method whenever we construct a piecewise linear approximation of an unknown function. This strategy is realized quite rarely (see [31,32]). This article exposes a numerical method relying on finite element method in the space variables and a special iteration scheme in which the heat transfer coefficient is sought in the form of a finite segment of a series. The number of summands in this series depends on the number of measurements. Note that this method does not rely on reducing the problem to a control problem. We use a direct method based on theoretical considerations exposed below. There is an existence and uniqueness theorem in this problem [30] with boundary observation points and theoretically a solution can be obtained by successive approximations. The process is convergent locally in time. This idea is realized numerically. The problems when the observation points lie insider the domain are generally ill-posed. But we show some examples in this case as well. The results of numerical experiments are quite satisfactory in both cases.

## 1. Preliminaries

In this section we describe notations and expose theoretical results which form the base for our constructions. The Sobolev and Hölder spaces are denoted conventionally (see [33–36]). Given the interval $J = (0, T)$, the symbols $W_p^{s,r}(Q) = W_p^s(J; L_p(G)) \cap L_p(J; W_p^r(G))$ and $W_p^{s,r}(S) = W_p^s(J; L_p(\Gamma)) \cap L_p(J; W_p^r(\Gamma))$ stand for the anisotropic Sobolev spaces. Let

$(u,v)=\int_G u(x)v(x)dx$. All function spaces and the coefficients in (1) – (3) are assumed to be real. By a norm of a vector we mean the sum of the norms of its coordinates. We say that the boundary $\Gamma$ of the given domain $G$ is of the class $C^s$ $(s\geq 1)$, if, for any point $x_0\in\Gamma$, there exists a neighborhood $Y_{x_0}$ (a coordinate neighborhood) of this point and a coordinate system $y$ (a local coordinate system) obtained by a rotation and a translation of the origin from the original coordinate system such that the $y_n$-axis is directed along the inward normal to $\Gamma$ at $x_0$ and the equation of the part $Y_{x_0}\cap\Gamma$ of the boundary has the form $y_n=\gamma(y')$, $\gamma(0)=0$, $|y'|<\delta$, $y'=(y_1,\dots,y_{n-1})$, where $\gamma\in C^s(\overline{B'_\delta})$ $(B'_\delta=\{y':|y'|<\delta\})$, $G\cap Y_{x_0}=\{y:|y'|<\delta, 0<y_n-\gamma(y')<\delta_1\}$, $(\mathbb{R}^n\setminus G)\cap Y_{x_0}=\{y:|y'|<\delta,-\delta_1<y_n-\gamma(y')<0\}$. The numbers $\delta$ and $\delta_1$ are fixed for the domain $G$, and without loss of generality we assume that $\delta_1>(M+1)\delta$, where $M$ is the Lipschitz constant of the function $\gamma$. Fix the parameter $\delta$ with the above properties. We take $\delta$ to be sufficiently small so that $\overline{B_\delta(b_i)}\cap\overline{B_\delta(b_j)}=\emptyset$ for $i\neq j$, $i,j=1,2,\dots,r$, where $B_\delta(b_i)$ is the ball of radius $\delta$ centered at $b_i$. Denote $Q^\tau=(0,\tau)\times G$, $G_\delta=\cup_i B_\delta(b_i)\cap G$, $\Gamma_\delta=\Gamma\cap\overline{G_\delta}$, $S_\delta=(0,T)\times\Gamma_\delta$, $S^\tau=(0,\tau)\times\Gamma$. Let

$$\Gamma\in C^2,\quad \Gamma_\delta\in C^3. \tag{4}$$

We use straightening of the boundary, i. e. the transformation $z_n=y_n-\gamma(y')$, $z'=y'$, with $y$ as a local coordinate system at $b_i$. We have that $x=x(y)=x_0+Ay$, with $A$ as an orthogonal matrix. Under condition (4), the transformation $x=x(y(z))=x^i(z)$ belongs to the class $C^3$ (i. e. $x^i(z)\in C^3(\overline{U})$, with $U=\{z:|z'|<\delta,0<z_n<\delta_1\}$). Denote $Q_0^\tau=(0,\tau)\times U$, $Q_0=(0,T)\times U$, $S_0^\tau=(0,\tau)\times B'_\delta$, and $S_0=(0,T)\times B'_\delta$. Describe our conditions on the data. We assume that

$$a_i\in L_p(Q),\ a_{kl}\in C(\overline{Q}),\ \beta_0,a_{kl}|_\Gamma\in W_p^{s_0,2s_0}(S),\ s_0=1/2-1/2p, p>n+2, \tag{5}$$

$$a_i\in L_\infty(0,T;W_p^1(G_\delta)),\ a_{kl}\in L_\infty(0,T;W_\infty^1(G_\delta)), \tag{6}$$

where $i=0,1\dots,n,\ k,l=1,\dots,n$,

$$u_0(x)\in W_p^{2-\frac{2}{p}}(G),\ f\in L_p(Q),\quad g\in W_p^{s_0,2s_0}(S), \tag{7}$$

Construct the functions $\varphi_i(x)\in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_i(x)=1$ in $B_{\delta/2}(b_i)$ and $\varphi_i(x)=0$ in $\mathbb{R}^n\setminus B_{3\delta/4}(b_i)$.

Let $Y_{b_i}$ be the coordinate neighborhood for $b_i\in\Gamma$. We require that

$$\nabla_{z'}\varphi_i g(t,x^i(z',0))\in W_p^{s_0,2s_0}(S_0),\ \nabla_{z'}\varphi_i f(t,x^i(z))\in L_p(Q_0),\quad i=1,2,\dots,r, \tag{8}$$

$$\nabla_{z'}\varphi_i u_0(x^i(z))\in W_p^{2-2/p}(U),\ \nabla_{z'}a_{kl}(t,x^i(z',0)),$$
$$\nabla_{z'}\beta_0(t,x^i(z',0))\in W_p^{s_0,2s_0}(S_0)\ \ (k,l=1,2\dots,n,\ i\leq r). \tag{9}$$

Note that conditions (8), (9) are independent of the local coordinate system $y$ and the coordinate system $z$.

Suppose that the functions $\Phi_i(t,x)$ possess the properties

$$\Phi_i\in W_p^{s_0,2s_0}(S),\ \nabla_{z'}\Phi_i(t,x^j(z',0))\in W_p^{s_0,2s_0}(S_0),\ i,j=1,2,\dots,r. \tag{10}$$

Denote by $\Phi(t)$ the matrix with entries $\phi_{ij} = \Phi_j(t, b_i)$ $(i, j = 1, 2, \dots, r)$ and assume that

$$|\psi_i(t)| \geq \delta_1,\ |det\,\Phi| \geq \delta_1 > 0\ \forall t \in [0, T],\ \psi_i \in W_p^{s_1}(0, T),\ u_0(b_i) = \psi_i(0), \tag{11}$$

where $\delta_1$ is a positive constant and $i = 1, 2, \dots, r$. Take the first of the equalities (2) at the point $(0, b_j)$. We have that

$$Bu_0(b_j) = \frac{\partial u_0(b_j)}{\partial N} + \beta(0, b_j)u_0(b_j) = g(0, b_j) - \beta_0(0, b_j)u_0(b_j),\ j = 1, \dots, r. \tag{12}$$

These equalities is a system relatively the quantities $\beta_i(0)$ which can be written as

$$\sum_{i=1}^{r} \beta_i(0)\Phi_i(0, b_j) = \frac{1}{u_0(b_j)}(g(0, b_j) - \frac{\partial u_0(b_j)}{\partial N} - \beta_0(0, b_j)u_0(b_j)),\ j = 1, \dots, r. \tag{13}$$

In view of (11), the consistency condition at $t = 0$ can be written as follows:

$$\frac{\partial u_0(x)}{\partial N} + (\beta(0, x) + \beta_0(0, x))u_0(x) = g(0, x)\ \forall x \in \Gamma, \tag{14}$$

where the constants $\beta_j(0)$ are solution to system (13).

**Theorem 1.** *Suppose that conditions* (4) – (11), (14) *hold. Then on some segment* $[0, \tau_0]$ *there exists a unique solution to problem* (1) – (3) *such that* $u \in W_p^{1,2}(Q^\tau)$, $\beta_i(t) \in W_p^{s_0}(0, \tau_0)$ $(i = 1, 2, \dots, r)$ *and* $\nabla_{z'}\varphi_i u(x^i(z)) \in W_p^{1,2}(Q_0^\tau)$, $i = 1, \dots, r$. *The solution depends continuously on the data.*

The arguments of the proof of this theorem (see [30]) allow to construct numerical algorithms of constructing an approximate solution. Let $\vec{\alpha} = (\beta_1, \beta_2, \dots, \beta_r)^T$. Solving problem (1), (2), we construct the mapping $\vec{\alpha} \to u(\vec{\alpha})$. Taking $x = b_j$ and using (3), we obtain the system

$$\sum_{i=1}^{r} \beta_i(t)\Phi_i(t, b_j) = \frac{1}{\psi_j(t)}(g(t, b_j) - \frac{\partial u(t, b_j)}{\partial N} - \beta_0(t, b_j)\psi_j(t)),\ j = 1, \dots, r. \tag{15}$$

which can be written in the form

$$\vec{\beta} = \Phi^{-1}\vec{F},\quad F_i = \frac{1}{\psi_j(t)}(g(t, b_j) - \frac{\partial u(t, b_j)}{\partial N} - \beta_0(t, b_j)\psi_j(t)),\ j = 1, \dots, r. \tag{16}$$

In operator form this equation can be written as follows:

$$\vec{\alpha} = R(\vec{\alpha}), \tag{17}$$

with $R$ as some operator. We could not prove in [30] that this operator is a contraction. But we proved that we can replace the functions $\dfrac{\partial u(t, b_j)}{\partial N}$ with some other functions constructed with the help of the function $u(\alpha)$ and the differential equation such that the operator constructed is a contraction in some Banach space and a solution can be found by successive approximations. We use a close idea in order to construct the numerical algorithm below.

## 2. Numerical Algorithm

Describe a numerical algorithm. We consider problem (1) – (3) in the case of $n = 2$ and the rectangle $G = (0, X) \times (0, Z)$. We take the equation

$$\begin{gathered} Mu = u_t - Lu = u_t - div(c(t,x)\nabla u) + b(x,t)\nabla u + a(t,x)u = f, \\ b(t,x) = (b_1(x,t), b_2(x,t))^T, \nabla u = (\tfrac{\partial u}{\partial x_1}, \tfrac{\partial u}{\partial x_2})^T, \end{gathered} \tag{18}$$

where $c = diag(c_1, c_2)$ is a diagonal matrix and $(t,x) \in Q = G \times (0,T)$. The overdetermination conditions are as follows:

$$u(y_i, t) = \psi_i(t), i = 1, 2, \ldots, r,\ y_i \in \Gamma_0 = \{(x_1, 0) :\ x_1 \in (0, X)\}. \tag{19}$$

Let $\Gamma = \partial G$, $S_0 = (0,T) \times \Gamma_0$. The initial and boundary value conditions are rewritten as follows:

$$u|_{t=0} = u_0(x),\ c_2 u_{x_2} - (\beta + \beta_0)u|_{S_0} = g(t, x_1),\ u|_{S \setminus S_0} = 0. \tag{20}$$

We have the consistency condition

$$u_0(x)|_{\Gamma \setminus \Gamma_0} = 0. \tag{21}$$

On the first step, we employ the finite element method. Let $\beta = \sum_{j=1}^{r} \alpha_j(t)\Phi_j(x_1)$, where the functions $\alpha_i$ are unknowns and the functions $\Phi_j$ are given functions. We have that $y_i = (x_1^i, 0)$. Given a triangulation of $G$ and the corresponding basis $\{\varphi_i\}_{i=1}^N$, the nodes are denoted by $\{b_i\}$. An approximate solution is representable as

$$v = \sum_{i=1}^{N} C_i(t)\varphi_i(x), \quad \varphi_i|_{\Gamma \setminus \Gamma_0} = 0,\ \varphi_i(b_j) = \delta_{ij},$$

where $\delta_{ij}$ stands for the Kronecker symbol. For convenience, we assume that the points $y_i$ agree with the nodes $b_1, \ldots, b_r$. The functions $C_i$ are a solution to the system

$$R_0 \vec{C}_t + R_1(t)\vec{C} = \vec{F} + \vec{f},\ \vec{C} = (C_1, C_2, \ldots, C_N)^T, \tag{22}$$

where

$$\vec{F} = (-\int_0^X \beta(t, x_1)v(t, x_1, 0)\varphi_1(x_1, 0)\, dx_1, \ldots, -\int_0^X \beta(t, x_1)v(t, x_1, 0)\varphi_N(x_1, 0)\, dx_1)^T,$$

and the coordinates of $\vec{f}$ are of the form

$$f_i = \int_G f(t,x)\varphi_i(x)\, dx - \int_0^X g(t, x_1)\varphi_i(x_1, 0)\, dx_1,$$

$R_0$ and $R_1$ are matrices with the respective entries $r_{ij} = (\varphi_i, \varphi_j) = \int_G \varphi_i(x)\varphi_j(x)\, dx$, and

$$\begin{aligned} R_{jk} = {} & (c_1(t,x)\varphi_{kx_1}, \varphi_{jx_1}) + (c_2(t,x)\varphi_{kx_2}, \varphi_{jx_2}) + \\ & + (b(t,x)\nabla\varphi_k, \varphi_j) + (a(t,x)\varphi_k, \varphi_j) + \int_0^X \beta_0(t, x_1)\varphi_k(x_1, 0)\varphi_j(x_1, 0)\, dx_1. \end{aligned}$$

We have $\vec{C}(0) = \vec{C}_0 = (u_0(b_1), \dots, u_0(b_N))$. We look for a solution to system (22) by a finite difference method. Given the step $\tau = T/M$ in time, replace equation (22) with the system

$$R_0\frac{\vec{C}_{i+1} - \vec{C}_i}{\tau} + A_{i+1}\vec{C}_{i+1} = \vec{F}_{i+1} + \vec{f}_{i+1}, \;\; \vec{C}_i = (C_i^1, \dots, C_i^N)^T, \; i = 0, 1, 2, \dots, M-1, \quad (23)$$

where $C_i^k \approx C_k(\tau i)$, $\vec{F}_i \approx \vec{F}(\tau i)$, $\;\; \vec{f}_i = \vec{f}(\tau i)$, $A_i = R_1(\tau i)$. The coordinates of the vector $\vec{F}_{i+1}$ are written as follows:

$$F_{i+1}^k = -\sum_{j=1}^{r}\alpha_{i+1}^j\Big(\sum_{l=1}^{r}\psi_{i+1}^l\int_0^X \Phi_j(x_1)\varphi_l(x_1,0)\varphi_k(x_1,0)\,dx_1 + \\ + \sum_{l=r+1}^{N} C_{i+1}^l\int_0^X \Phi_j(x_1)\varphi_l(x_1,0)\varphi_k(x_1,0)\,dx_1\Big), \; k = 1, 2, \dots, r, \quad (24)$$

$$F_{i+1}^k = -\sum_{j=1}^{r}\alpha_i^j\Big(\sum_{l=1}^{N} C_i^l\int_0^X \Phi_j(x_1)\varphi_l(x_1,0)\varphi_k(x_1,0)\,dx_1\Big), \; k > r, \quad (25)$$

where $\vec{\alpha}_i = (\alpha_i^1, \dots, \alpha_i^r)^T$ ($\vec{\alpha}_i \approx \vec{\alpha}(\tau i)$), $\vec{\Psi}(t) = (\psi_1, \psi_2, \dots, \psi_r)^T$, and $\vec{\Psi}_i = \vec{\Psi}(\tau i)$. Rewrite (23) in the form

$$(R_0 + \tau A_{i+1})\vec{C}_{i+1} = \tau\vec{F}_{i+1} + \tau\vec{f}_{i+1} + R_0\vec{C}_i, \; i = 0, 1, 2, \dots, M-1. \quad (26)$$

Define the matrices $R_0^l$ ($l = 1, 2, 3, 4$) with entries $r_{jk}^1 = r_{jk}$ ($j = 1, 2, \dots, r$, $k = 1, 2, \dots, r$), $r_{jk}^2 = r_{jk}$ ($j = 1, 2, \dots, r$, $k = r+1, \dots, N$), $r_{jk}^3 = r_{jk}$ ($j = r+1, \dots, N$, $k = 1, 2, \dots, r$), and $r_{jk}^4 = r_{jk}$ ($j = r+1, \dots, N$, $k = r+1, \dots, N$). Respectively, define the matrix $A_{i+1}^i$ ($l = 1, 2, 3, 4$) with entries $a_{jk}^1 = R_{jk}$ ($j = 1, 2, \dots, r$, $k = 1, 2, \dots, r$), $a_{jk}^2 = R_{jk}$ ($j = 1, 2, \dots, r$, $k = r+1, \dots, N$), $a_{jk}^3 = R_{jk}$ ($j = r+1, \dots, N$, $k = 1, 2, \dots, r$), $a_{jk}^4 = R_{jk}$ ($j = r+1, \dots, N$, $k = r+1, \dots, N$). Also, we introduce the matrix $B_{i+1}^0$ with entries $b_{kj}^{i+1} = -\Big(\sum_{l=1}^r \psi_{i+1}^l\int_0^X \Phi_j(x_1)\varphi_l(x_1,0)\varphi_k(x_1,0)\,dx_1 + \sum_{l=r+1}^N C_{i+1}^l\int_0^X \Phi_j(x_1)\varphi_l(x_1,0)\varphi_k(x_1,0)\,dx_1\Big)$ ($j = 1, 2, \dots, r$, $k = 1, \dots, r$) and the matrix $B_{i+1}^1$ with entries $b_{kj}^{i+1} = -\Big(\sum_{l=1}^N C_i^l\int_0^X \Phi_j(x_1)\varphi_l(x_1,0)\varphi_k(x_1,0)\,dx_1\Big)$ ($j = 1, \dots, r$, $k = r+1, r+2, \dots, N$). Also, introduce the vectors $\vec{f}_{i+1}^0 = (f_{i+1}^1, \dots, f_{i+1}^r)^T$, $\vec{f}_{i+1}^1 = (f_{i+1}^{r+1}, \dots, f_{i+1}^N)^T$, $\vec{C}_{i+1}^1 = (C_{i+1}^{r+1}, \dots, C_{i+1}^N)^T$. In this case system (26) can be rewritten in the form

$$(R_0^1 + \tau A_{i+1}^1)\vec{\psi}_{i+1} + (R_0^2 + \tau A_{i+1}^2)\vec{C}_{i+1}^1 = \tau B_{i+1}^0\vec{\alpha}_{i+1} + \tau\vec{f}_{i+1}^0 + R_0^1\vec{\psi}_i + R_0^2\vec{C}_i^1,$$

$$(R_0^3 + \tau A_{i+1}^3)\vec{\psi}_{i+1} + (R_0^4 + \tau A_{i+1}^4)\vec{C}_{i+1}^1 = \tau B_{i+1}^1\vec{\alpha}_i + \tau\vec{f}_{i+1}^1 + R_0^3\vec{\psi}_i + R_0^4\vec{C}_i^1,$$

where $i = 0, 1, 2, \dots, M-1$, or in the form

$$\tau\vec{\alpha}_{i+1} = (B_{i+1}^0)^{-1}[(R_0^1 + \tau A_{i+1}^1)\vec{\psi}_{i+1} + (R_0^2 + \tau A_{i+1}^2)\vec{C}_{i+1}^1 - \tau\vec{f}_{i+1}^0 - R_0^1\vec{\psi}_i - R_0^2\vec{C}_i^1], \quad (27)$$

$$\vec{C}_{i+1}^1 = (R_0^4 + \tau A_{i+1}^4)^{-1}[-(R_0^3 + \tau A_{i+1}^3)\vec{\psi}_{i+1} + \tau B_{i+1}^1\vec{\alpha}_i + \tau\vec{f}_{i+1}^1 + R_0^3\vec{\psi}_i + R_0^4\vec{C}_i^1], \quad (28)$$

where $i = 0, 1, 2, \ldots, M-1$. Thus, an approximate solution is written as

$$v(\tau(i+1), x) = \sum_{k=1}^{r} \psi_{i+1}^{k} \varphi_k + \sum_{k=r+1}^{N} C_{i+1}^{k} \varphi_k(x), \ i = 0, 1, \ldots, M-1,$$

Define the initial data. We have $C_0^k = u_0(b_k)$ for $k = 1, \ldots, N$. The numbers $\vec{\alpha}_0^k = \tilde{C}_0^k$ are solutions to the system

$$\sum_{i=1}^{r} \alpha_0^i \Phi_i(b_k)\psi_k(0) = c_2(0, b_k)u_{0x_2}(b_k) - \beta_0(0, b_k)\psi_k(0) - g(0, b_k), \ k = 1, \ldots, r.$$

An analog of condition (11) is the condition

$$\det \{\Phi_i(b_k)\}_{k,i=1}^{r} \neq 0, \ \psi_i(t) \neq 0 \ \forall t, i. \tag{29}$$

For $i = 0$, the right-hand side of system (28) is known and we can find the vector $\vec{C}_1^1$, using system (27) we determine $\vec{\alpha}_1$. Repeating the arguments we can find the vectors $\vec{C}_{i+1}^1$ for all $i$ and respectively the vectors $\vec{\alpha}_i$. The convergence is improved if we use the idea of the predictor-corrector method. Assume that the vectors $\vec{\alpha}_i$ and $\vec{C}_i$ are given. Using (28) and (27) we can determine $\vec{\alpha}_{i+1}$ and $\vec{C}_{i+1}$. Assign $\vec{\alpha}_i^0 := \vec{\alpha}_{i+1}$. Inserting this vector in (28) instead of $\vec{\alpha}_i$ we can find the vector $\vec{C}_{i+1}$ again and assign $\vec{C}_{i+1}^1 := \vec{C}_{i+1}$. Using (27) we can determine $\vec{\alpha}_{i+1}$ and assign $\vec{\alpha}_{i+1}^1 := \vec{\alpha}_{i+1}$. Inserting this vector in (28) instead of $\vec{\alpha}_i$, we again determine $\vec{C}_{i+1}$ and assign $\vec{C}_{i+1}^2 := \vec{C}_{i+1}$. Using (28), we determine $\vec{\alpha}_{i+1}$ and assign $\vec{\alpha}_{i+1}^2 := \vec{\alpha}_{i+1}$. The process is repeated until $|\vec{\alpha}_{i+1}^j - \vec{\alpha}_{i+1}^{j-1}| \leq \varepsilon$, where $\varepsilon > 0$ is a small given number. If the latter holds for some $j$, we assign

$$\vec{C}_{i+1} := \vec{C}_{i+1}^j, \ \vec{\alpha}_{i+1} := \vec{\alpha}_{i+1}^j.$$

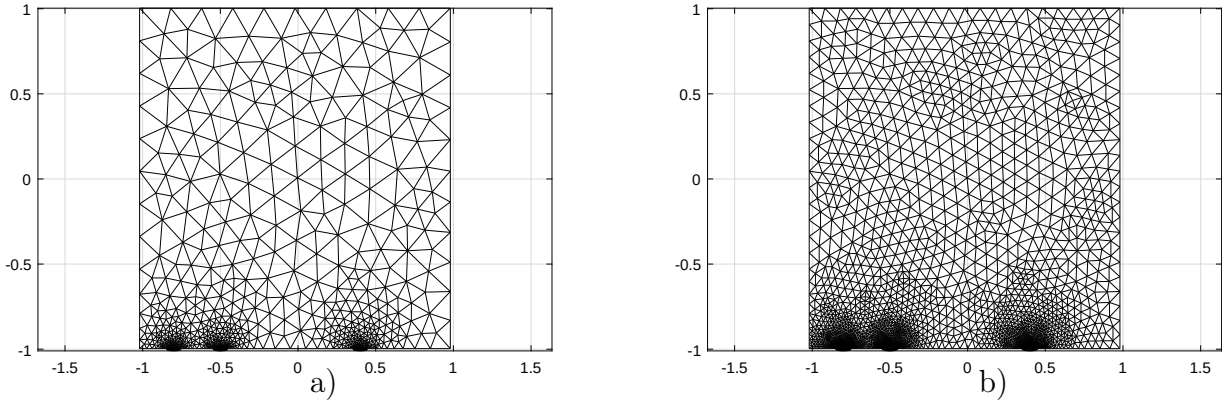
Note that the problem is nonlinear and generally we can not guarantee that the matrix $B_{i+1}^0$ remains nondegenerate for all $i$ (see (27)) and condition (29) ensures that $det\, B_{i+1}^0 \neq 0$ only locally in time. To improve the convergence in a singular case, we can use the Tikhonov regularization replacing the matrix $(B_{i+1}^0)^{-1}$ in (27) with the matrix $((B_{i+1}^0)^* B_{i+1}^0 + \varepsilon I)^{-1}(B_{i+1}^0)^*$, where $\varepsilon > 0$ is the parameter of regularization and $I$ is the identity matrix. The same algorithm can be used in the case of interior points $\{y_i\}_{i=1}^r$.

## 3. Results of Numerical Experiments

In this section we analyze the results of numerical experiments. The characteristics of the computer are as follows: the processor Intel(R) Core(TM) i5-9500F CPU @ 3.00GHz, 16.00 GB RAM, the 64-digit operating system Windows 10 Pro.

As a result of calculations, we obtain approximate values of a solution $(u(x_1, x_2, t), \ \alpha_1(t), \ldots, \alpha_r(t)$ to problem (18) – (21) at the points $(t_1, t_2, \ldots t_N)$. Here the point $(x, y)$ belongs to the rectangle $(-1, 1)^2$ and $T = 1$. We use meshes with different number of nodes. Two of them with the number of nodes $N_i$ $(i = 1, 2)$ are shown in Fig. 1.

All numerical experiments contain unknown functions $u$, $\alpha_i$, the boundary conditions, $\delta$ (the noise percentage divided by 100), $\varepsilon$ (the estimate for the norms $|\vec{\alpha}_{i+1}^j - \vec{\alpha}_{i+1}^{j-1}|$), the coefficients $b_1, b_2, a$, and the right-hand sides $f$.




a) b)

**Fig. 1**. Meshes a) $N_1 = 799$; b) $N_2 = 3069$

We take $\varepsilon = 10^{-3}$, $\tau = \Delta t = T/M$, $M = 100$ and use the following data.

In the first case,

the solution is $u(x,t) = xt + x^2yt + (x+1)\cdot(y+1)$;

the initial data are $u|_{t=0} = (x+1)\cdot(y+1)$;

the additional information is $\psi_i(t) = u(y_i,t)$; ($y_i$ are observation points);

the coefficients are $a = (t+1)(x^2+1)$, $b_1 = tx^3$, $b_2 = 0$, $c_1 = x+1$, $c_2 = x^2+1$;

the right-hand side is $f = xt + tyx^2 + (x+1)\cdot(y+1) - 1 - x\cdot(t^2+2) - y\cdot(t-2)^3)$;

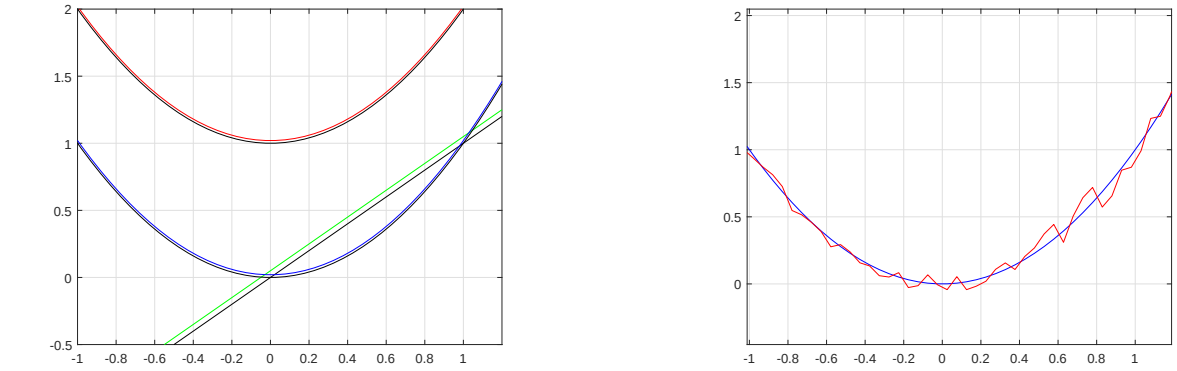
the coefficient $\beta = 1 + xt + t^2 + x^2t^2$;

$\Phi_1 = 1$, $\Phi_2 = x$, $\Phi_3 = x^2$ and $\alpha_1 = 1 + t^2, \alpha_2 = t, \alpha_3 = t^2$;

the Robin boundary condition is $g = -t^3x^3 - t^3x - t^2x^3 - 2t^2x^2 - t^2x - t^2 + tx^4 - 2tx + x^3 + x^2$;

the addition information is given at the observation points: $x_{j_1} = (x_1,\ y_1) = (-0,5,-1)$; $x_{j_2} = (x_2,\ y_2) = (0,4,-1)$; $x_{j_3} = (x_3,\ y_3) = (-0,8,-1)$.

We use mesh with the number of nodes $N_1 = 799$. The result of calculation is shown in Fig. 2 a).



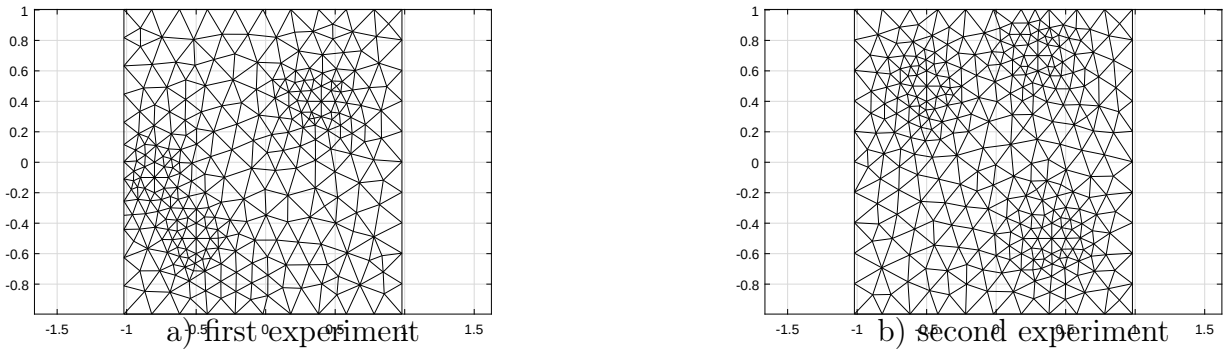

a) b) $\delta = 0,1$, $i = 3$

**Fig. 2**. Results of recovering $\alpha_i$

We can see that the initial curves and the results of calculations almost coincide. Next, we consider the same data but perturb the data with some random variable (Fig. 2 b). The perturbations of the overdetermination data at the moments of time $\Delta tk$, $k = 1,2,\dots,M$

($\Delta t$ is a step in time) are defined as follows: $\tilde{\psi}_{it}(\Delta tk) = \psi_{it}(\Delta tk)(1 + \delta(2\sigma_{ik} - 1))$, where the numbers $\sigma_{ik} \in [0, 1]$ are determined using the random number generator of Matlab (the function rand).

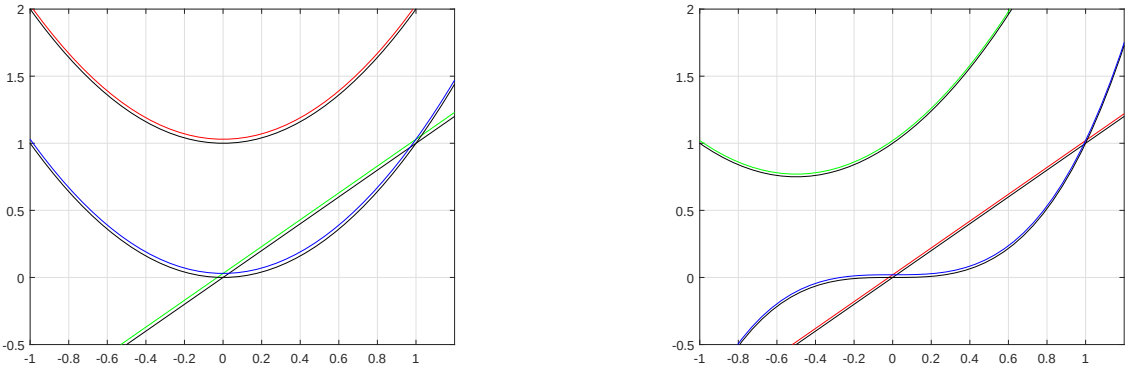
Next, we present another example with interior observation points. We take the meshes (Fig. 3).



a) first experiment b) second experiment

**Fig. 3**. Meshes a) $N_1 = 323$; b) $N_2 = 301$

We use the first mesh ($N_1 = 323$) and the above data but the additional information is given at the following observation points (Fig. 4 a): $x_{j_1} = (x_1,\ y_1) = (-0,5, -0,5)$; $x_{j_2} = (x_2,\ y_2) = (0,4, 0,4)$; $x_{j_3} = (x_3,\ y_3) = (-0,8, -0,1)$.



a) first experiment b) second experiment

**Fig. 4**. Results of recovering the functions $\alpha_i$

In the second case we take the mesh with $N_2 = 301$ and consider the case of the following interior observation points: $x_{j_1} = (x_1,\ y_1) = (0,4, -0,5)$; $x_{j_2} = (x_2,\ y_2) = (0,3, 0,7)$; $x_{j_3} = (x_3,\ y_3) = (-0,5, 0,5)$. The data are as follows:

the solution is $u(x,t) = xt + x^2yt + (x+1)\cdot(y+1)$;

the initial data are $u|_{t=0} = (x+1)\cdot(y+1)$;

the additional information is $\psi_i(t) = u(y_i, t)$; ($y_i$ are observation points)

the coefficients are $a = (t+1)(x^2+1)$, $b_1 = tx^3$, $b_2 = 0$, $c_1 = (x+2)\cdot(x+1)$, $c_2 = 4/(10x^2+1)$;

the right-hand side is $f = xt + tyx^2 + (x+1)\cdot(y+1) - 1 - x\cdot(t^2+2) - y\cdot(t-2)^3)$.

The coefficient $\beta = 1 + xt + t^2 + x^3t^3$. If $\Phi_1 = 1$, $\Phi_2 = x$, $\Phi_3 = x^3$ then $\alpha_1 = 1 + t^2$, $\alpha_2 = t$, $\alpha_3 = t^3$.

the Neumann boundary conditions are $g = c_2 u_y - \beta u|_{y=0} = c_2(tx^2 + (x+1)) - \beta(xt + x + 1) = -1 - x - t^4x^4 + t^2(-1 - x - x^2) + (4(1+x))/(1+10x^2) + t^3(-x - x^3 - x^4) + t(2/5 - 2x - x^2 - 2/(5(1+10x^2)))$.

First calculations (Fig. 4 b) are made in the case of the data without noise, i. e., $\delta = 0$. Next, we add 5, 10, 15 and 25 percent random noise (Fig. 5).

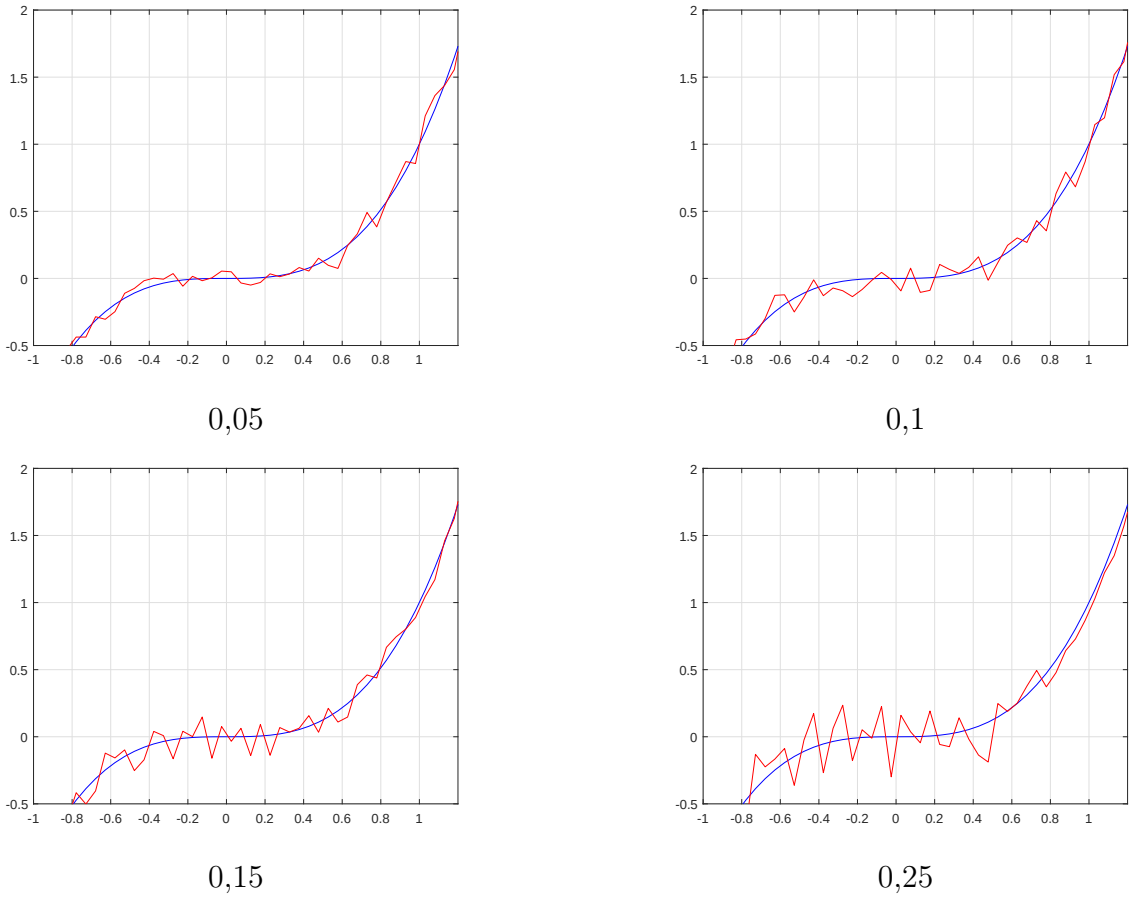



**Fig. 5**. Results of calculations with a random error for $\alpha_2$

Summing up, we can say that the algorithm shows good sustainability because as we can see in experiments in the second case (Fig. 5), we can get correct information especially using some smoothing algorithms. Decreasing the variable $\varepsilon$ leads to an increase in the computation time $\tau$, but does not lead to a significant increase in accuracy. We also see that the dependence of the calculation time on the time step is inversely proportional.

Finally, we present summary table of experiments with the last data and different $\delta$ and $\varepsilon$. Denote by $\tau$ the time of the work of the algorithm in seconds. Introduce the error of the algorithm by the equality $\varepsilon_0 = \max\limits_m |\vec{\alpha}_m - \vec{\alpha}_m(\Delta tm)|$, where $m = 1, 2, \ldots, M$.

## Conclusion

Using theoretical results on well-posedness of the problem we construct a numerical algorithm for recovering of the heat transfer coefficient with the use of point observations of the temperature. It is based on the conventional methods (in our case the finite element

**Table**

Results of numerical experiments for the second mesh

| No exp. | Grid | $\delta$ | $\varepsilon$ | $\varepsilon_0$ | $\tau$ |
|---|---|---|---|---|---|
| 1 | $N_2$ | 0 | $10^{-3}$ | 0,0045 | 5,34 |
| 2 | $N_2$ | 0 | $10^{-4}$ | 0,0051 | 5,95 |
| 3 | $N_2$ | 0 | $10^{-5}$ | 0,0052 | 6,13 |
| 4 | $N_2$ | 0 | $10^{-6}$ | 0,0061 | 6,67 |
| 5 | $N_2$ | 0,1 | $10^{-3}$ | 0,0132 | 5,8 |
| 6 | $N_2$ | 0,1 | $10^{-4}$ | 0,0134 | 5,89 |
| 7 | $N_2$ | 0,1 | $10^{-5}$ | 0,0145 | 6,23 |
| 8 | $N_2$ | 0,1 | $10^{-6}$ | 0,0167 | 6,55 |
| 9 | $N_2$ | 0,2 | $10^{-3}$ | 0,0262 | 6,16 |
| 10 | $N_2$ | 0,2 | $10^{-4}$ | 0,0279 | 6,74 |
| 11 | $N_2$ | 0,2 | $10^{-5}$ | 0,0278 | 7,02 |
| 12 | $N_2$ | 0,2 | $10^{-6}$ | 0,0291 | 7,43 |

method and difference schemes). The results of numerical experiments are presented. The obtained results reveal the accuracy, efficiency, and robustness of the proposed algorithm. It is stable under random perturbations of the data.

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# ВОССТАНОВЛЕНИЕ КОЭФФИЦИЕНТА ТЕПЛОПЕРЕДАЧИ ПО РЕЗУЛЬТАТАМ ИЗМЕРЕНИЙ ТЕМПЕРАТУРЫ

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Теория обратных задач используется, чтобы восстановить коэффициент теплопередачи в задачах теплопроводности, используя замеры температуры на границе. Численное решение основано на методе конечных элементов по пространственным переменным, методе конечных разностей по времени и специальной итерационной схемы для определения коэффициента теплопередачи на каждом временном слое. Коэффициент теплопередачи ищется в виде конечного отрезка ряда с неизвестными коэффициентами Фурье, зависящими от времени. Алгоритм решения опирается на теоретические результаты, утверждающие, что задача корректна и сводится к операторному уравнению со сжимающим оператором. Результаты численных экспериментов подтверждают, что задача действительно корректна. Полученные результаты показывают точность, эффективность и надежность предложенного алгоритма. Они устойчивы к случайным возмущениям.

*Ключевые слова: обратная задача; коэффициент теплопередачи; параболическое уравнение; тепломассоперенос.*

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