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STOCHASTIC LEONTIEFF TYPE EQUATIONS AND MEAN DERIVATIVES OF STOCHASTIC PROCESSES

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We view the Leontieff type stochastic differential equations as a special sort of Ito stochastic differential equations, in which the left-hand side contains a degenerate constant linear operator and the right-hand side has a non-degenerate constant linear operator. On the right-hand side there is also a summand with a term depending only on time. Its physical meaning is the incoming signal into the device described by the operators mentioned above. In the papers by A.L. Shestakov and G.A. Sviridyuk the dynamical distortion of signals is described by such equations. Transition to stochastic differential equations arise where it is necessary to take into account the interference (noise). Note that the investigation of solutions of such equations requires the use of derivatives of the incoming signal and the noise of any order. In this paper for differentiation of noise we apply the machinery of the so-called Nelson's mean derivatives of stochastic processes. This allows us to avoid using the machinery of the theory of generalized functions. We present a brief introduction to the theory of mean derivatives, investigate the transformation of the equations to canonical form and find formulae for solutions in terms of Nelson's mean derivatives of Wiener process.

Keywords: mean derivative, current velocity, Wiener process, Leontieff type equation.

Introduction

In the papers [1, 2] some approach to investigation on dynamically distorted signals is suggested that is based on Leontieff type differential equations. Further development of this approach requires taking interference (noise) into account that yields the transition to Stochastic Differential Equations. Here the correspondent stochastic differential equation takes the form

$$\tilde{L}\xi(t) = \tilde{M}\int_0^t \xi(s)ds + f(t) + B\tilde{w}(t),$$

where $\tilde{L}$ is a degenerate $n \times n$ matrix, $\tilde{M}$ and B are non-degenerate $n \times n$ matrices, $\xi(t)$ is an n-dimensional stochastic process, $f(t)$ is a smooth n-dimensional vector-function and $\tilde{w}(t)$ is a Wiener process in $\mathbb{R}^n$. The physical meaning of these objects is as follows: $f(t)$ is the signal incoming into the device described by the matrices $\tilde{L}$ and $\tilde{M}$, while $B\dot{\tilde{w}}(t)$ (where $\dot{\tilde{w}}(t)$ is ≪derivative≫ of Wiener process, i.e., white noise) describes the noise (interference).

The equations of such sort are called the Leontieff type stochastic differential equations.

The features of Leontieff type equations require dealing with the derivatives of $f(t)$ and $w(t)$ of any order. In paper [3] in the simplest case where the incoming signal is absent B is the unit matrix and the equation has been reduced to canonical form, the so called current velocities (symmetric mean derivatives) of the Wiener process are involved for describing the derivatives of the Wiener process. As a result some physically reasonable analytical formulae for the solutions are obtained.

The notion of mean derivatives was introduced by E. Nelson [4, 5, 6] for the needs of the so-called Nelson's stochastic mechanics (a version of quantum mechanics). Later a lot of applications of mean derivatives to some other branches of science were found. The investigation of Leontieff type stochastic differential equations is a new field of application of mean derivatives. Note that by the general ideology of the theory of Nelson's mean derivatives the current velocities are natural analogues of physical velocity of deterministic processes.

In this paper by the use of current velocities we investigate the general situation and do not suppose the equation to be already reduced to canonical form. Some constructions connected to reducing the equations to canonical form are announced in [7].

An alternative approach to investigation of Leontieff type stochastic equations, also based on the use of current velocities, is suggested in [8].

Besides the Introduction the paper contains three Sections. The first is devoted to basic preliminary fact from the theory of mean derivatives necessary for the purpose of this article. In Section 2 we investigate the transition of Leontieff type stochastic differential equations to canonical form. In Section 3 we find formulae for the solutions of equations under consideration.

Throughout the paper we use Einstein's summation convention with respect to shared upper and lower indices.

We refer the reader to [9, 10] for details on the machinery of mean derivatives.

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1. Preliminaries on the mean derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is L_1-random variable for all t.

Every stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, determines three families of σ-subalgebras of σ-algebra $\mathcal{F}$:
(i) the «past» $\mathcal{P}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;
(ii) the «future» $\mathcal{F}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le l$;
(iii) the «present» («now») $\mathcal{N}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by the mapping $\xi(t)$.
All families are supposed to be complete, i.e., containing all sets of probability 0.

For convenience we denote the conditional expectation of $\xi(t)$ with respect to $\mathcal{N}_t^\xi$ by $E_t^\xi(\cdot)$.

The ordinary («unconditional») expectation is denoted by E.

Strictly speaking, almost surely (a.s.) the sample paths of $\xi(t)$ are not differentiable for almost all t. Thus its «classical» derivatives exist only in the sense of generalized functions. To avoid using the generalized functions, following Nelson (see, e.g., [4, 5, 6]) we give

Definition 1. *(i) The forward mean derivative $D\xi(t)$ of $\xi(t)$ with respect t is an L_1-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t}\right) \tag{1}$$

where the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0 and $\Delta t > 0$.

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ with respect t is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\right) \tag{2}$$

where the conditions and the notation are the same as in (i).

Note that mainly $D\xi(t) \neq D_*\xi(t)$, but if, say, $\xi(t)$ a.s. has smooth sample paths, these derivatives evidently coinside.

From the properties of conditional expectation (see [11]) it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$Y^0(t,x) = \lim_{\Delta t \to +0} E(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} | \xi(t) = x)$$

$$Y^0_*(t,x) = \lim_{\Delta t \to +0} E(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} | \xi(t) = x) \tag{3}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t,\xi(t))$ and $D_*\xi(t) = Y^0_*(t,\xi(t))$.

The derivatives introduced in Definition 1, is a particular case of the objects defined as follows. Let $x(t)$ and $y(t)$ be L_1-stochastic processes in $\mathbb{R}^n$, given on $(\Omega, \mathcal{F}, \mathsf{P})$. Introduce the y-forward mean derivative of $x(t)$ by the formula

$$D^y x(t) = \lim_{\Delta t \to +0} E^y_t(\frac{x(t+\Delta t) - x(t)}{\Delta t}) \tag{4}$$

and the y-backward mean derivative of $x(t)$ by the formula

$$D^y_* x(t) = \lim_{\Delta t \to +0} E^y_t(\frac{x(t) - x(t-\Delta t)}{\Delta t}) \tag{5}$$

where the limits must exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$.

Recall that a process $\xi(t)$ is called martingale (in our case–with respect its «past» $\mathcal{P}^\xi_t$), if for every time instants $0 \leq s < t \leq l$ the relation $E(\xi(t) \mid \mathcal{P}^\xi_s) = \xi(s)$ takes place.

Lemma 1. *Let $\xi(t)$ be a martingale with respect to its «past» $\mathcal{P}^\xi_t$. Then $D\xi(t) = 0$.*

Proof. By the properties of the conditional expectation $E^\xi_t(E(\cdot \mid \mathcal{P}^\xi_t)) = E^\xi_t(\cdot)$. Then $E^\xi_t(\xi(t+\Delta t) - \xi(t)) = E^\xi_t(E(\xi(t+\Delta t) - \xi(t) \mid \mathcal{P}^\xi_t)) = E^\xi_t(\xi(t) - \xi(t)) = 0$. □

Definition 2. *The derivative $D_S = \frac{1}{2}(D + D_*)$ is called the symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called the anti-symmetric mean derivative.*

Consider the vector fields $v^\xi(t,x) = \frac{1}{2}(Y^0(t,x) + Y^0_*(t,x))$ and $u^\xi(t,x) = \frac{1}{2}(Y^0(t,x) - Y^0_*(t,x))$.

Definition 3. *$v^\xi(t) = v^\xi(t,\xi(t)) = D_S\xi(t)$ is called current velocity of $\xi(t)$;*
$u^\xi(t) = u^\xi(t,\xi(t)) = D_A\xi(t)$ is called osmotic velocity of $\xi(t)$.

For stochastic processes the current velocity is a direct analogue of ordinary physical velocity of deterministic processes (see, e.g., [4, 5, 6, 9, 10]). The osmotic velocity measures how fast the «randomness» grows up.

By $w(t)$ we denote the Wiener process. Recall that $w(t)$ is a Wiener process (in our case, with respect to its own «past» $\mathcal{P}^w_t$), if

1) its sample paths are a.s. continuous in t;

2) $w(t)$ is a square integrable martingale with respect to $\mathcal{P}^w_t$ such that $w(0) = 0$ and $E((w(t) - w(s))^2 | \mathcal{P}^w_t) = t - s$ for $t \geq s$.

Well-known Levi's theorem says that in addition $w(t)$ has stationary independent Gaussian increments and satisfies the equalities:

$$E(w(t) - w(s)) = 0, \qquad E((w(t) - w(s))^2) = t - s$$

for $t \geq s$. In the other words, the increment $w(t) - w(s)$ for $t \geq s$ is independent of $\mathcal{P}^w_s$ and has the same distribution as $w(t-s)$. Note that the probabilistic density $\rho^w(t,x)$ of $w(t)$ in $\mathbb{R}^n$ takes the form

$$\rho^w(t,x) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{x^2}{2t}}. \tag{6}$$

Recall that the sample paths of $w(t)$ are a.s. non-differentiable for almost all t and on every arbitrarily small time inervals they a.s. have infinite variation. Thus, the derivatives of $w(t)$ in usual sense exists only as a generalized function.

Below we often deal with the processes of the form

$$\xi(t) = \xi_0 + \int_0^t \beta(s)ds + w(t) \tag{7}$$

where $w(t)$ is a Wiener process. For such processes the above-mantioned «physical» properties of current and osmotic velocities become clear from the following propositions.

Denote by $\rho^\xi(t,x)$ the density of process (7) with respect to Lebesgue measure λ on $[0,l]\times\mathbb{R}^n$. This means that for every continuous inntegrable function $f(t,x)$ on $[0,l]\times\mathbb{R}^n$ the following equality takes place:

$$\int_{[0,l]\times\mathbb{R}^n} f(t,x)\rho^\xi(t,x)d\lambda = \int_{\Omega\times[0,l]} f(t,\xi(t))d\mathsf{P}dt.$$

Lemma 2. *For process (7) in $\mathbb{R}^n$ the vector field $u^\xi(t,x)$ is represented in the form*

$$u^\xi(t,x) = \frac{1}{2}\mathrm{grad}\, log\, \rho^\xi(t,x). \tag{8}$$

Lemma 3. *For process (7) in $\mathbb{R}^n$ the vector field $v^\xi(t,x)$ and the density $\rho^\xi(t,x)$ satisfy the equation of continuity*

$$\frac{\partial \rho^\xi(t,x)}{\partial t} = -\mathrm{div}(\rho^\xi v^\xi). \tag{9}$$

The proofs of Lemmas 2 and 3 in the form convenient for us, can be found in [9, 10].

For processes of more general type the above Lemmas can be generalized as follows.

Lemma 4. [12] *Let $\xi(t)$ satisfies the Ito equation $\xi(t) = \int_0^t a(s,\xi(s))ds + \int_0^t A(s,x)dw(s)$. Then*

$$u^\xi(t,x) = \frac{1}{2}\frac{\frac{\partial}{\partial x^j}(\alpha^{ij}\rho^\xi(t,x))}{\rho^\xi(t,x)}\frac{\partial}{\partial x^i} \tag{10}$$

where (α^{ij}) is the matrix of the operator AA^*.

Proof. Let f be an arbitrary smooth function on R^n with compact support. Note that $f(\xi(t))$ is $\mathcal{N}^\xi_t$-measurable. Hence

$$E\Big(f(\xi(t))E^\xi_t(\int_{t-\Delta t}^t A(t,\xi(t))dw(t))\Big) = E\Big(f(\xi(t))\int_{t-\Delta t}^t A(t,\xi(t))dw(t)\Big).$$

Since $f(\xi(t-\Delta t))$ and $\int_{t-\Delta t}^t A(t,\xi(t))dw(t)$ are independent and $E\int_{t-\Delta t}^t A(t,\xi(t))dw(t) = 0$, we have

$$E(f(\xi(t))(\int_{t-\Delta t}^t A(t,\xi(t))dw(t)) = E\left((f(\xi(t)) - f(\xi(t-\Delta t)))(\int_{t-\Delta t}^t A(t,\xi(t))dw(t))\right).$$

By the Ito formula $f(\xi(t)) - f(\xi(t-\Delta t)) = \int_{t-\Delta t}^{t}(df \cdot a(s,\xi(s)))ds + \frac{1}{2}\int_{t-\Delta t}^{t} tr\, f''(\xi(s))ds + \int_{t-\Delta t}^{t}(df \cdot A(s,\xi(s)))dw(s)$ (by $\cdot$ we denote the coupling of 1-forms and vectors). Thus

$$E\left(f(\xi(t))\int_{t-\Delta t}^{t} A(t,\xi(t))dw(t)\right) = E\Big(\int_{t-\Delta t}^{t}(df \cdot a(s,\xi(s)))A(s,\xi(s))dsdw(s)$$

$$+\frac{1}{2}\int_{t-\Delta t}^{t} tr\, f''(\xi(s))(A(s,\xi(s)),A(s,\xi(s)))dsdw(s) + \int_{t-\Delta t}^{t}(df \cdot A(s,\xi(s)))A(s,\xi(s))ds\Big).$$

The first two integrals on the right-hand side equal zero. Calculations in coordinates show that $(df \cdot A)A = df \cdot (AA^*)$.

On the other hand,

$$\int_0^T E\Big(f(\xi(t))u^\xi(t,\xi(t))\Big)dt = -\frac{1}{2}\int_0^T E\left(f(\xi(t))\lim_{\Delta t\to+0} E_t^\xi\Big(\frac{\int_{t-\Delta t}^{t} A(s,\xi(s))dw(s)}{\Delta t}\Big)\right)dt =$$

$$-\frac{1}{2}\int_0^T E(df \cdot AA^*)dt = -\frac{1}{2}\int_{R^n\times[0,T]} df \cdot AA^* \cdot \rho^\xi dt\wedge\Lambda = \frac{1}{2}\int_{R^n\times[0,T]} f \cdot d(AA^* \cdot \rho^\xi)dt\wedge\Lambda =$$

$$\frac{1}{2}\int_{R^n\times[0,T]} f\frac{d(AA^*\cdot\rho^\xi)}{\rho^\xi}\rho^\xi dt\wedge\Lambda = \frac{1}{2}\int_0^T E\Big(f\frac{d(AA^*\cdot\rho^\xi)}{\rho^\xi}\Big)dt = \frac{1}{2}\int_0^T E\Big(f\frac{\frac{\partial}{\partial x^j}(\alpha^{ij}\rho^\xi)}{\rho^\xi}\frac{\partial}{\partial x^i}\Big)dt.$$

Since this is valid for an arbitrary f as above, this means that $u^\xi = \frac{1}{2}\frac{d(AA^*\cdot\rho^\xi)}{\rho^\xi} = \frac{1}{2}\frac{\frac{\partial}{\partial x^j}(\alpha^{ij}\rho^\xi)}{\rho^\xi}\frac{\partial}{\partial x^i}$.

□

An alternative proof of Lemma 4 can be found in [12].

Let A as above be constant and non-degenerate. Then the matrix $(\alpha_{ij}) = (\alpha^{ij})^{-1}$ is well-posed and it can be considered as the matrix of the new inner product in R^n. In this case we obtain

Corollary 1.

$$u^\xi(t,x) = \frac{1}{2}Grad\log\rho^\xi(t,x) = Grad\log\sqrt{\rho^\xi(t,x)} \tag{11}$$

where Grad denotes the gradient with respect the inner product with the matrix (α_{ij}).

Indeed, if A is constant, (α_{ij}) is constant as well, and formula (10) takes the form

$$u^\xi(t,x) = \frac{1}{2}\frac{\frac{\partial}{\partial x^j}(\alpha^{ij}\rho^\xi(t,x))}{\rho^\xi(t,x)}\frac{\partial}{\partial x^i} = \frac{1}{2}\alpha^{ij}\frac{\frac{\partial}{\partial x^j}(\rho^\xi(t,x))}{\rho^\xi(t,x)}\frac{\partial}{\partial x^i} =$$

$$\frac{1}{2}Grad\log\rho^\xi(t,x) = Grad\log\sqrt{\rho^\xi(t,x)}.$$

We are using formulae (8) and (11) below.

Now consider the autonomous smooth field of non-degenerate linear operators $A(x):\mathbb{R}^n\to\mathbb{R}^n$, $x\in\mathbb{R}^n$ (i.e., $(1,1)$-tensor field on $\mathbb{R}^n$). Let $\xi(t)$ be a diffusion process in which the integrand under Itô integral is of the form $A(\xi(t))$. Then its diffusion coefficient $A(x)A^*(x)$ is a smooth field of symmetric positive definite matrices $\alpha(x) = (\alpha^{ij}(x))$ ($(2,0)$-tensor field on $\mathbb{R}^n$). Since all these matrices are non-degenerate and smooth, there exist the smooth field of converse symmetric and positive definite matrices (α_{ij}). Hence this field can be used as a new Riemannian $\alpha(\cdot,\cdot) = \alpha_{ij}dx^i\otimes dx^j$ on $\mathbb{R}^n$. The volume form of this metric has the form $\Lambda_\alpha = \sqrt{\det(\alpha_{ij})}dx^1\wedge dx^2\wedge\cdots\wedge dx^n$.

Denote by $\rho^\xi(t,x)$ the probability density of random element $\xi(t)$ with respect to the volume form $dt\wedge\Lambda_\alpha = \sqrt{\det(\alpha_{ij})}dt\wedge dx^1\wedge dx^2\wedge\cdots\wedge dx^n$ on $[0,T]\times\mathbb{R}^n$, i.e., for every continuous bounded function $f:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ the relation

$$\int\limits_0^T E(f(t,\xi(t)))dt = \int\limits_0^T\left(\int\limits_\Omega f(t,\xi(t))d\mathsf{P}\right)dt = \int\limits_{[0,T]\times\mathbb{R}^n} f(t,x)\rho^\xi(t,x)dt\wedge\Lambda_\alpha.$$

holds.

Lemma 5. *For $v^\xi(t,x)$ and $\rho^\xi(t,x)$ the equation of continuity takes the form*

$$\frac{\partial\rho^\xi(t,x)}{\partial t} = -Div(v^\xi(t,x)\rho^\xi(t,x)), \tag{12}$$

where Div denotes the divergence with respect to Riemannian metric $\alpha(\cdot,\cdot)$.

Proof. Here by Λ_E we denote the form $dx^1\wedge\cdots\wedge dx^n$. So, $\Lambda=\sqrt{\det(\alpha_{ij})}\Lambda_E$.

Recall that $Div(\rho^\xi v^\xi) = *^{-1}d((\rho^\xi v^\xi)\rfloor\Lambda)$ where $\rfloor$ is the interrior product of vector $(\rho^\xi v^\xi)$ and n-form Λ. But $(\rho^\xi v^\xi)\rfloor\Lambda = \sqrt{\det(\alpha_{ij})}\sum_{i=1}^n(\rho^\xi v^\xi)^i dx^1\wedge\cdots\wedge dx^{i-1}\wedge dx^{i+1}\wedge\cdots\wedge dx^n$ and so $Div(\rho^\xi v^\xi) = \frac{(\rho^\xi v^\xi)^i}{\sqrt{\det(\alpha_{ij})}}\frac{\partial\sqrt{\det(\alpha_{ij})}}{\partial x^i} + \frac{\partial(\rho^\xi v^\xi)^i}{\partial x^i}$.

Specify a smooth function $f(t,x)$ with compact support. By df we denote the differential with respect to spacial coordinates: $df=\frac{\partial f}{\partial x^i}dx^i$. Note that by coordinate calculations we get

$$\int_{[s,t]\times R^n}\Big(df\cdot(\rho^\xi v^\xi(\tau,\xi(\tau)))\Big)d\tau\wedge\Lambda =$$

$$\int_{[s,t]\times R^n}\Big(df\cdot(\rho^\xi v^\xi(\tau,\xi(\tau)))\sqrt{\det(\alpha_{ij})}\Big)d\tau\wedge\Lambda_E =$$

$$-\int_{[s,t]\times R^n}\Big(f(\tau,x)\left[\sqrt{\det(\alpha_{ij})}\frac{\partial(\rho^\xi v^\xi)^i}{\partial x^i} + (\rho^\xi v^\xi)^i\frac{\partial\sqrt{\det(\alpha_{ij})}}{\partial x^i}\right]\Big)d\tau\wedge\Lambda_E =$$

$$-\int_{[s,t]\times R^n}\Big(f(\tau,x)\left[\frac{\partial(\rho^\xi v^\xi)^i}{\partial x^i} + \frac{(\rho^\xi v^\xi)^i}{\sqrt{\det(\alpha_{ij})}}\frac{\partial\sqrt{\det(\alpha_{ij})}}{\partial x^i}\right]\sqrt{\det(\alpha_{ij})}\Big)d\tau\wedge\Lambda_E =$$

$$-\int_{[s,t]\times R^n}\Big(f(t,x)\left[\frac{\partial(\rho^\xi v^\xi)^i}{\partial x^i} + \frac{(\rho^\xi v^\xi)^i}{\sqrt{\det(\alpha_{ij})}}\frac{\partial\sqrt{\det(\alpha_{ij})}}{\partial x^i}\right]\Big)d\tau\wedge\Lambda =$$

$$-\int_{[s,t]\times R^n}\Big(f(\tau,x)Div(\rho^\xi v^\xi)\Big)d\tau\wedge\Lambda.$$

By the Ito formula

$$E\Big(f(t,\xi(t))-f(s,\xi(s))\Big) = E\Big(\int_s^t\frac{\partial f}{\partial\tau}d\tau + \int_s^t df\cdot Y^0(\tau,\xi(\tau))d\tau + \frac{1}{2}\int_s^t tr\,f''(A,A)d\tau\Big)$$

and by the backward Ito formula

$$E\Big(f(t,\xi(t))-f(s,\xi(s))\Big) = E\Big(\int_s^t\frac{\partial f}{\partial\tau}d\tau + \int_s^t df\cdot Y^0_*(\tau,\xi(\tau))d\tau - \frac{1}{2}\int_s^t tr\,f''(A,A)d\tau\Big).$$

Hence,

$$E\Big(f(t,\xi(t)) - f(s,\xi(s))\Big) = E\Big(\int_s^t \frac{\partial f}{\partial \tau}d\tau + \int_s^t df \cdot v^\xi(\tau,\xi(\tau))d\tau\Big).$$

But

$$E\Big(\int_s^t \frac{\partial f}{\partial \tau}d\tau + \int_s^t df \cdot v^\xi(\tau,\xi(\tau))d\tau\Big) = \int_{[s,t]\times R^n}\Big(\frac{\partial f}{\partial \tau}\rho^\xi + [df\cdot(\rho^\xi v^\xi(\tau,\xi(\tau)))]\Big)d\tau\wedge\Lambda =$$

$$\int_{[s,t]\times R^n}\frac{\partial}{\partial\tau}\Big((f(\tau,x)\rho^\xi\Big)d\tau\wedge\Lambda - \int_{[s,t]\times R^n}\Big(f(\tau,x)\frac{\partial\rho^\xi}{\partial\tau}\Big)d\tau\wedge\Lambda$$

$$-\int_{[s,t]\times R^n}\Big(f(\tau,x)Div(\rho^\xi v^\xi)\Big)d\tau\wedge\Lambda =$$

$$E\Big(f(t,\xi(t)) - f(s,\xi(s))\Big) - \int_{[s,t]\times R^n}\Big(f(\tau,x)\frac{\partial\rho^\xi}{\partial\tau}\Big)d\tau\wedge\Lambda - \int_{[s,t]\times R^n}\Big(f(\tau,x)Div(\rho^\xi v^\xi)\Big)d\tau\wedge\Lambda.$$

Thus $\int_{[s,t]\times R^n}\Big(f(\tau,x)\frac{\partial\rho^\xi}{\partial\tau}\Big)d\tau\wedge\Lambda + \int_{[s,t]\times R^n}\Big(f(\tau,x)Div(\rho^\xi v^\xi)\Big)d\tau\wedge\Lambda = 0$. Since this is valid for an arbitrary $f(t,x)$ as above, this means that $\frac{\partial\rho^\xi}{\partial\tau} = -Div(\rho^\xi v^\xi)$.

□

An alternative proof of can be found in [6].

Since $w(t)$ is a martingale, $Dw(t) = 0$, $t \in [0,l)$ (see above).

Lemma 6. *[See, e.g., [9, 10]] For $t \in (0,l]$ the equality $D_*w(t) = \frac{w(t)}{t}$ holds.*

Proof. From the definition of osmotic velocity $u^w(t,w(t))$ it follows that $D_*w(t) = -2u^w(t,w(t))$. Since $\rho^w(t,X)$ is given by formula (6), from formula (8) it follows that $u^w(t,x) = -\frac{1}{2}\cdot\frac{x}{t}$. Thus, $D_*w(t) = \frac{w(t)}{t}$.

□

Corollary 2. $D_Sw(t) = \frac{w(t)}{2t}$.

Let us turn to calculation of the higher orders mean derivatives of $w(t)$. Taking into account the system of notation from [9, 10], we look for the k derivative as D^w, D^w_* or D^w_S (see (4) and (5)) of the $(k-1)$-th derivatives. This notation emphasizes that we always use the σ-algebra «present» of $w(t)$.

Lemma 7. *[See, e.g., [9, 10]] (i) $D^w\frac{w(t)}{t} = -\frac{w(t)}{t^2}$ for $t \in (0,l)$.*

(ii) $D^w_*\frac{w(t)}{t} = 0$ for $t\in(0,l]$.

(iii) $D^w_S\frac{w(t)}{t} = -\frac{w(t)}{2t^2}$ for $t \in (0,l]$.

Proof. Indeed,

$$D^w\frac{w(t)}{t} = (\frac{d}{dt}\frac{1}{t})w(t) + \frac{1}{t}Dw(t) = -\frac{w(t)}{t^2}$$

and

$$D^w_*\frac{w(t)}{t} = (\frac{d}{dt}\frac{1}{t})w(t) + \frac{1}{t}D_*w(t) = -\frac{w(t)}{t^2} + \frac{w(t)}{t^2} = 0.$$

Assertion (iii) follow from the last two formulae.

□

Lemma 8. *(i)* $D^w(\frac{w(t)}{t^k}) = -k\frac{w(t)}{t^{k+1}}$;
(ii) $D^w_*(\frac{w(t)}{t^k}) = -(k-1)\frac{w(t)}{t^{k+1}}$
(iii) $D^w_S(\frac{w(t)}{t^k}) = -\frac{2k-1}{2}\frac{w(t)}{t^{k+1}}$.

Proof.
(i) $D^w(\frac{w(t)}{t^k}) = \frac{d}{dt}\frac{1}{t^k}w(t) + \frac{1}{t^k}Dw(t) = -k\frac{w(t)}{t^{k+1}} + 0 = -k\frac{w(t)}{t^{k+1}}$

(ii) $D^w_*(\frac{w(t)}{t^k}) = \frac{d}{dt}\frac{1}{t^k}w(t) + \frac{1}{t^k}D_*w(t) = -k\frac{w(t)}{t^{k+1}} + \frac{1}{t^k}\frac{w(t)}{t} = -(k-1)\frac{w(t)}{t^{k+1}}$;

(iii) From the last two formulae we obtain that $D^w_S(\frac{w(t)}{t^k}) = -\frac{2k-1}{2}\frac{w(t)}{t^{k+1}}$.

□

Lemma 9. [1] *For integer $k \geq 2$*

$$D^k_S w(t) = (-1)^{k-1}\cdot\frac{\prod_{i=1}^{k-1}(2i-1)}{2^k}\cdot\frac{w(t)}{t^k}.$$

This formula is proved by induction starting from the assertions of Corollary 2, Lemma 7 (iii) and Lemma 8 (iii).

2. Leontieff type stochastic equations and their canonical form

As it is mentioned in the Introduction, the stochastic differential equation of Leontieff type is a stochastic differential equation in $\mathbb{R}^n$ of the form $\tilde{L}\xi(t) = \tilde{M}\int_0^t \xi(s)ds + \int_0^t f(s))ds + B\tilde{w}(t)$, where $\xi(t)$ is a random and $f(t)$ is a deterministic n-dimensional vectors, $\tilde{L}$, $\tilde{M}$ and B are $n\times n$ matrices, where $\tilde{L}$ is degenerate (has zero determinant) while $\tilde{M}$ and B are non-degenerate and $\tilde{w}(t)$ is a Wiener process. Their physical meaning is a follows: $f(t)$ is an incoming signal into the device described by operators $\tilde{L}$ and $\tilde{M}$, $B\dot{\tilde{w}}$ where $\dot{\tilde{w}}(t)$ is white noise, is interference, and $\xi(t)$ is outgoing signal. The vector-function $f(t)$ is supposed to be smooth.

If the sheaf $\tilde{M}+\lambda\tilde{L}$ is regular, one can apply the Kronecker-Weierstrass transformation and reduce the matrices $\tilde{L}$ and $\tilde{M}$ to the quasi-diagonal form (see [13]). This transformation is described by a pair of linear non-degenerate operators (matrices) that we denote by $A = (a^i_j)$ and A_R. The conjugate to A operator is denoted by A^*. In quasi-diagonal form, under appropriate numeration of basis vectors, in the matrix $L = A\tilde{L}A_R$ first along diagonal there are Jordan boxes with zeros on diagonal, and the last matrix along diagonal is the unit one. In $M = A\tilde{M}A_R$ in the lines corresponding to Jordan boxes, there is the unit matrix and the last block along diagonal is a certain non-degenerate matrix. In the next section, for the sake of convenience, we present the matrices L and M in explicit form.

Denote by $(\cdot,\cdot)$ the standard inner product (Euclidean metric) in $\mathbb{R}^n$. Recall that the Wiener process $\tilde{w}(t)$ is Gaussian with mean value 0 and covariation matrix tI, where I is the unit matrix, i.e, with density (6) with respect to the volume form of Euclidean metric $(\cdot,\cdot)$.

Introduce the matrix $C = AB$. Since the matrices A and B are non-degenerate, C is non-degenerate as well and such is also $CC^* = ABB^*A^*$. Hence the inverse matrix $(CC^*)^{-1} = C^{*-1}C^{-1}$ is well-posed. Thus (see [14]), $C\tilde{w}(t)$ is also Gaussian with mean value 0 and covariation matrix tCC^* and so, with density

$$\rho^{C\tilde{w}}(t,x) = ((2\pi t)^{-n/2}\Delta^{-1/2})exp(\frac{-((CC^*)^{-1}x,x)}{2t}) \tag{13}$$

[1] Note the misprint in this formula in [3] where mistakenly 2^{k-1} instead of 2^k is set in the denominator.

with respect to the same volume form, where Δ is determinant of CC^*.

Introduce the new inner product (Eucliden metric) $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$ by formula $\langle X,Y\rangle = ((CC^*)^{-1}X,Y)$.

Theorem 1. *(i) For every vectors X and Y in $\mathbb{R}^n$ the identity $\langle CX,CY\rangle = (X,Y)$ holds. (ii) The process $w(t) = C\tilde{w}(t)$ is a Wiener process in $\mathbb{R}^n$ with Euclidean metric $\langle\cdot,\cdot\rangle$.*

Proof. Recall that $(CC^*)^{-1} = C^{*-1}C^{-1}$. Then

$$\langle CX,CY\rangle = (C^{*-1}C^{-1}CX,CY) = (C^{-1}CX,C^{-1}CY) = (X,Y).$$

The volume form of the metric $\langle\cdot,\cdot\rangle$ differs from that of $(\cdot,\cdot)$ by the coefficient $\Delta^{-1/2}$, i.e., the density of $C\tilde{w}(t)$ with respect to the volume form of $\langle\cdot,\cdot\rangle$ takes the form

$$((2\pi t)^{-n/2})exp(\frac{-((CC^*)^{-1}x,x)}{2t}) = ((2\pi t)^{-n/2})exp(\frac{-\langle x,x\rangle}{2t}). \tag{14}$$

Obviously the other properties of Wiener process are satisfied for $C\tilde{w}(t)$ in $\mathbb{R}^n$ with metric $\langle\cdot,\cdot\rangle$.

□

Let $e_1,\ldots,e_n$ be a natural orthonormal basis for $\mathbb{R}^n$ with $(\cdot,\cdot)$.

Corollary 3. *$Ce_1,\ldots,Ce_n$ is an orthonormal basis for $\mathbb{R}^n$ with $\langle\cdot,\cdot\rangle$.*

Corollary 4. *Introduce $\eta(t) = A_R^{-1}\xi(t)$. In $\mathbb{R}^n$ with $\langle\cdot,\cdot\rangle$ the Leontieff type stochastic equation takes the form $L\eta(t) = \int_0^t M\eta(s)ds + \int_0^t Af(s)ds + w(t)$.*

Taking into account formula (11), we see that the expression of the current velocity for $w(t)$ contains $Grad(C^{-1}x,C^{-1}x)$, where $Grad$ is the gradient with respect to the inner product $\langle\cdot,\cdot\rangle$.

Lemma 10. *$d\langle x,x\rangle = d(C^{-1}x,C^{-1}x) = 2C^{*-1}C^{-1}x$, where d is the exterior differential.*

Lemma 11. *$Grad\langle x,x\rangle = Grad(C^{-1}x,C^{-1}x) = 2x$.*

The proof follows from the formula of lifting the indices

$$Grad(C^{-1}x,C^{-1}x) = CC^*d(C^{-1}x,C^{-1}x)$$

and from Lemma 10.

Hence, in $\mathbb{R}^n$ with $\langle\cdot,\cdot\rangle$ formulae for the current velocity and higher symmetric derivatives of Wiener process $w(t)$ have usual form as in Lemmas 6 – 9.

3. Solutions of Leontieff type stochastic equations

So (see Corollary 4), if the sheaf $\tilde{M} + \lambda\tilde{L}$ is regular, after the Kronecker-Weierstrass transformation the Leontieff type stochastic equation in $\mathbb{R}^n$ with $\langle\cdot,\cdot\rangle$ takes the form

$$L\eta(t) = \int_0^t M\eta(\tau)d\tau + \int_0^t Af(\tau)d\tau + w(t), \tag{15}$$

where $\eta(t) = A_R^{-1}\xi(t)$,

$$L = A\tilde{L}A_R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \qquad (16)$$

and

$$M = A\tilde{M}A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^{n-q}_{n-q} & a^{n-q}_{n-q+1} & \dots & a^{n-q}_{n} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^{n-q+1}_{n-q} & a^{n-q+1}_{n-q+1} & \dots & a^{n-q+1}_{n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^{n}_{n-q} & a^{n}_{n-q+1} & \dots & a^{n}_{n} \end{pmatrix}. \qquad (17)$$

Everywhere below we deal with equation (15) in $\mathbb{R}^n$ with $\langle\cdot,\cdot\rangle$.

It is clear (cf. (7)), that here for simplicity the initial value in (15) is supposed to be $\xi(0) = 0$. Note that the solutions that we construct below, cannot satisfy this condition since they are ill-posed at $t = 0$. That is why we approximate the solutions by processes that satisfy zero initial condition but become solutions only after a certain, a priori given and arbitrarily small positive time instant $t_0 > 0$ (see below).

Remark 1. Rewrite (15) in the form $L\eta(t) - M\int_0^t \eta(s)ds - A\int_0^t f(s)ds = w(t)$. We see that «present» for the process $L\eta(t) - M\int_0^t \eta(s)ds - A\int_0^t f(s)ds$ coincides with the «present» for $w(t)$. Thus we use the latter σ-algebra for calculation of mean derivatives< i.e., we apply to (15) the derivatives D^w, D^w_* or D^w_S. Note that the solutions found below, are measurable with respect to the «present» of $w(t)$ for every t.

Taking into account the structure of matrices (16) and (17), it is clear that (15) is decomposed into several independent systems of equations. The one «at the bottom» corresponds to the unit diagonal part of L and the last block of non-degenerate matrix in M. Denote the latter matrix by K, and by $\zeta(t)$ the vector of dimension $q+1$ constructed from the last $q+1$ coordinates of

$\eta(t)$. Then $\zeta(t)$ is described by the equation

$$\zeta(t) = K\int_0^t \zeta(s)ds + \int_0^t Af(\tau)d\tau + w(t) \tag{18}$$

in $\mathbb{R}^{q+1}$. Here $w(t)$ is a $q+1$-dimensional Wiener process constructed from the last $q+1$ coordinates of $w(t)$ in $\mathbb{R}^n$ and $Af(t)$ is a $q+1$-dimensional vector constructed from the last $q+1$ coordinates of $Af(t)$. For (18) there is a well-known analytical formula of solutions: $\zeta(t) = \int_0^t e^{K(t-\tau)}Af(\tau)d\tau + \int_0^t e^{K(t-\tau)}dw(\tau)$.

The other systems correspond to the Jordan boxes in L and unit matrices, constructed from the lines and columns in M. As an example, we consider $(p+1)\times(p+1)$ matrix (Jordan box) N in the left upper corner of (16)

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and the corresponding unit matrix from (17). The other systems are quite analogous.

Denote by $(Af)_{(p+1)}$ the $(p+1)$-dimensional vector constructed from the first $p+1$ coordinates of Af, by $\eta_{(p+1)}(t)$ – the $(p+1)$-dimensional vector with coordinates $(\eta^1(t),\dots,\eta^{p+1}(t))$ constructed from the first $(p+1)$ coordinates of $\eta(t)$ and by $w_{(p+1)}(t)$ – the vector with coordinates $(w^1(t),\dots,w^{p+1}(t))$ constructed from the first $p+1$ coordinates of $w(t)$. It is clear that the coordinates of Af have the form $(Af)^i = \sum_{j=1}^n a^i_j f^j$. Then $\eta_{(p+1)}(t)$ is described by the equation

$$N\eta_{(p+1)}(t) = \int_0^t \left(\eta_{(p+1)}(s) + (Af)_{(p+1)}(s)\right) ds + w_{(p+1)}(t).$$

Written in coordinates, this system takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta^1(t) \\ \eta^2(t) \\ \vdots \\ \eta^p(t) \\ \eta^{p+1}(t) \end{pmatrix} = \begin{pmatrix} \int_0^t(\eta^1(s) + \sum_{j=1}^n a^1_j f^j)ds \\ \int_0^t(\eta^2(s) + \sum_{j=1}^n a^2_j f^j)ds \\ \vdots \\ \int_0^t(\eta^p(s) + \sum_{j=1}^n a^p_j f^j)ds \\ \int_0^t(\eta^{p+1} + \sum_{j=1}^n a^{p+1}_j f^j)ds \end{pmatrix} + \begin{pmatrix} w^1(t) \\ w^2(t) \\ \vdots \\ w^p(t) \\ w^{p+1}(t) \end{pmatrix}. \tag{19}$$

From the last equation of (19) we obtain

$$\int_0^t \eta^{p+1}(s)ds = -\int_0^t (\sum_{j=1}^n a^{p+1}_j f^j)ds - w^{p+1}(t). \tag{20}$$

Since the current velocity (symmetric mean derivative) corresponds to the physical velocity, from this equation we find $\eta^{p+1}(t)$ by applying the derivative D^w_S to both sides of the equality (see Remark 1). Obvious application of the mean derivatives D^w and D^w_* (and so D^w_S) to the integrals both in the left and the right-hand sides yields the same results: $\eta^{p+1}(t)$ and $\sum_{j=1}^n a^{p+1}_j f^j$, respectively. Thus we see that

$$\eta^{p+1}(t) = -\sum_{j=1}^n a^{p+1}_j f^j - D^w_S w^{p+1}(t) = -\sum_{j=1}^n a^{p+1}_j f^j - \frac{w^{p+1}(t)}{2t}. \tag{21}$$

From the last but one equation we obtain

$$\eta^{p+1}(t) = \int_0^t (\eta^p(s) + \sum_{j=1}^{n} a_j^p f^j) ds + w^p(t). \tag{22}$$

Applying the arguments analogous to the above ones, we derive

$$\eta^p(t) = D_S^w \eta^{p+1}(t) - \sum_{j=1}^{n} a_j^p f^j - D_S^w w^p(t).$$

Substituting the expression for $\eta^{p+1}(t)$ from (21) into the latter equality and using Lemma 7, we obtain

$$\eta^p(t) = -\sum_{j=1}^{n} a_j^{p+1} \frac{df^j}{dt} - \sum_{j=1}^{n} a_j^p f^j + \frac{w^{p+1}(t)}{4t^2} - \frac{w^p(t)}{2t}. \tag{23}$$

By complete analogy, for $1 \le i \le p$ we obtain the recurrent formula

$$\eta^i(t) = D_S^w \eta^{i+1}(t) - \sum_{j=1}^{n} a_j^i f^j - D_S^w w^i(t). \tag{24}$$

Taking into account Lemma 9 we derive from (24) the explicit expression for every $\eta^i(t)$, $1 \le i \le p$ in the form:

$$\eta^i(t) = -\sum_{k=i}^{p} \left(\sum_{j=1}^{n} a_j^{k+1} \frac{d^{k-i+1} f^j}{dt^{k-i+1}} \right) - \sum_{j=1}^{n} a_j^i f^j$$

$$+ \sum_{k=i+1}^{p+1} \left((-1)^{k-i+1} \frac{\prod\limits_{j=1}^{k-i} (2j-1)}{2^{k-i+1}} \frac{w^k(t)}{t^{k-i+1}} \right) - \frac{w^i(t)}{2t}. \tag{25}$$

Let us turn back to the question on zero initial values for solutions of system (19). From the definition of symmetric mean derivatives it clearly follows that they are well-posed only on open time-intervals since their construction involves both forward and backward time increments. Taking into account formula (25), one can easily see that the solutions constructed above, have the form of sums where some summands contain multipliers of $\frac{w^j(t)}{t^k}$, $k \ge 1$, type. So, the solutions tend to zero as $t \to 0$, i.e., at $t = 0$ the values do not exist.

A version of solving this problem is as follows. Specify an arbitrary small time instant $t_0 \in (0, l)$ and consider the function $t_0(t)$ by the formula

$$t_0(t) = \begin{cases} t_0 & \text{if} \quad 0 \le t \le t_0; \\ t & \text{if} \quad t_0 \le t. \end{cases}$$

In formulae (21), (23) and (24) replace the elements $\frac{w^j(t)}{t^k}$ by $\frac{w^j(t)}{(t_0(t))^k}$. After that the processes will take zero value at $t = 0$ but only for $t > t_0$ they will be the solutions of (15). Note that for two different time instants $t_0^{(1)}$ and $t_0^{(2)}$, for $t > max(t_0^{(1)}, t_0^{(2)})$ the values of corresponding solutions coincide a.s.

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СТОХАСТИЧЕСКИЕ УРАВНЕНИЯ ЛЕОНТЬЕВСКОГО ТИПА И ПРОИЗВОДНЫЕ В СРЕДНЕМ СЛУЧАЙНЫХ ПРОЦЕССОВ

Ю.Е. Гликлих, Е.Ю. Машков

Стохастические дифференциальные уравнения леонтьевского типа мы понимаем как специальный класс стохастических дифференциальных уравнений в форме Ито, у которых в левой части имеется вырожденный постоянный линейный оператор, а в правой части – невырожденный постоянный линейный оператор. Также в правой части имеется слагаемое, зависящее только от времени. Его физический смысл – входящий сигнал в устройство, описываемое указанными выше операторами. В статьях А.Л. Шестакова и Г.А. Свиридюка подобные уравнения использованы для описания динамически искаженных сигналов. Переход к стохастическим дифференциальным уравнениям возникает при необходимости учета помех. Отметим, что для исследования решений таких уравнений необходимо использовать производные произвольного порядка от сигнала и от помех. В этой статье для дифференцирования помех мы применяем аппарат так называемых производных в среднем по Нельсону от случайных процессов. Это позволяет при исследовании не использовать аппарат теории обобщенных функций. Мы даем краткое введение в теорию производных в среднем, исследуем преобразование уравнений к каноническому виду и находим формулы для решений в терминах производных в среднем винеровского процесса.

Ключевые слова: производная в среднем, текущая скорость, винеровский процесс, уравнение леонтьевского типа.

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