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DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH REGULAR LOCAL MATRIX PENCILS

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We dedicate this paper to the memory of Yuri E. Boyarintsev, who was one of the pioneers in discovering differential-algebraic equations.

In the projector based framework, any regular linear DAE features several continuous time-varying characteristic subspaces that are independent of construction technicalities, among them the so-called sum-subspaces. As it is well-known, the local matrix pencils of a higher-index time-varying linear DAE do not reflect the global structure of the DAE in general. We show that, on the given interval, the local matrix pencils of the DAE are regular and reflect the global DAE structure if several of these characteristic subspaces are time-invariant. We discuss practicable methods to check the time-invariance of these subspaces. The corresponding class of DAEs is related to the class of DAEs formerly introduced and discussed by Yuri E. Boyarintsev.

Keywords: time-varying DAEs; local matrix pencil; regularity.

Introduction

Inspired by the fundamental meaning of regular matrix pencils for linear constant coefficient differential-algebraic equations (DAEs), e.g. [6], several early approaches to DAEs

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{1}$$

with matrix coefficients $A(t), B(t) \in \mathbb{R}^{m,m}$ depending continuously on $t \in \mathcal{I}$, assume that the so-called *local matrix pencils* $\lambda A(t) + B(t)$, $t \in \mathcal{I}$, are regular.

For instance, in [4, Section 2.4], the DAE (1) is said to be *regular* if it has exclusively regular local matrix pencils on the given interval, which is motivated by the feasibility of integration methods the implicit Euler method, for instance.

If the local matrix pencils are regular and there is a number c such that $cA(t) + B(t)$ is nonsingular for all t, then, by substituting $x(t) = \exp(ct)\tilde{x}(t)$, the DAE (1) can be transformed into the special form

$$\underbrace{(cA(t) + B(t))^{-1}A(t)}_{\tilde{A}(t)}\tilde{x}'(t) + \tilde{x}(t) = (cA(t) + B(t))^{-1}\exp(-ct)q(t), \quad t \in \mathcal{I}. \tag{2}$$

In turn, the special form (2) serves as a vantage point for the use of the Drazin inverse, similarly as for time-invariant DAEs. Obviously, Y.E. Boyarintsev was motivated by this fact: In [1, Chapter 5], the above property appears as an essential ingredient of Boyarintsev's regularity notion for DAEs. Later on, this property generally accounts for regularity: In [2, Definition 3.7.1], the DAE (1) is said to be *regular on a compact interval* $\mathcal{I}$ if there is a value c such that $cA(t) + B(t)$ becomes nonsingular for all $t \in \mathcal{I}$.

More recent notions, e.g. [3, Section 2.1], also Section 2. below, do no longer tie regularity of DAEs to regular local matrix pencils. Nevertheless, the interest in DAEs showing regular local matrix pencils persists for different reasons.

After discussing general local DAE aspects in Section 1, in Section 2. we characterize a class of DAEs whose local pencils uniformly reflect the global DAE structure in terms of the projector based framework and we expose there relations to the special DAE class formerly introduced by Boyarintsev. This class is actually characterized by certain time-invariant subspaces. Finally, in Section 3, we consider possibilities to check the time-invariance of these subspaces.

1. Boyarintsev's Ω-condition

The class of linear DAEs showing regular local matrix pencils is broad and relevant for many applications. If $A(t)$ remains nonsingular on the interval $\mathcal{I}$, then the DAE (1) is actually a regular ordinary differential equation, which is also called regular index-0 DAE. Trivially, then all local matrix pencils are regular with index 0. Further, if the DAE (1) has differentiation index 1, then all its local matrix pencils are regular with index 1.
Furthermore, all DAEs in Hessenberg form of arbitrary size $\mu \in \mathbb{N}$ belong to this class, their differentiation index equals $\mu \in \mathbb{N}$, and the local matrix pencils are regular with index $\mu \in \mathbb{N}$, too [4, Section 2.4].

On the other hand, as it has been well-known for a long time (e.g. [7, 8, 4]), the local matrix pencils are not necessarily regular for higher-index DAEs (1). We illustrate this fact by the following two simple examples.

Example 1. The constant coefficient DAE

$$\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}\bar{x}'(t)+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\bar{x}(t)=q(t),\quad t\in\mathcal{I}:=[0,1],\tag{3}$$

is regular with Kronecker index 2 owing to the matrix pencil properties. By means of a regular transformation K with

$$\bar{x}(t)=K(t)x(t),\ K'(t)=H(t)K(t),\ t\in\mathcal{I},K(0)=I,\ H(t):=\begin{bmatrix}-1&0&0\\1&0&-1\\0&0&0\end{bmatrix},$$

the regular DAE (3) is transformed into the time-varying DAE

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}K(t)}_{A(t)}\,x'(t)+\underbrace{(\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}H(t)K(t)+K(t))}_{B(t)}\,x(t)=q(t),\quad t\in\mathcal{I}.\tag{4}$$

For arbitrary numbers c we arrive at

$$cA(t)+B(t)=\begin{bmatrix}c&0&0\\0&1&0\\1&c&0\end{bmatrix}K(t),\quad \det(cA(t)+B(t))=0,\quad t\in\mathcal{I},$$

which means that the local matrix pencils of the DAE (4) are uniformly singular. However, no doubt, regular transformations have to preserve basic DAE properties. The DAE (4) has regular tractability index 2 and differentiation index 2, and it inherits the solvability of (3).

Example 2. The time-varying DAE

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix} x'(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix} x(t) = q(t), \quad t \in \mathcal{I}, \tag{5}$$

has regular tractability index 3 and differentiation index 3 on each arbitrary interval $\mathcal{I}$ as well as the corresponding solvability properties, but all the local matrix pencils are singular.

We emphasize that there are also DAEs that have exclusively regular local matrix pencils, but do not show a regular solvability behavior. The DAE in the next example is classified as nonregular with index 0 by [10, Chapter 10].

Example 3. The time-varying DAE

$$\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix} x'(t) + x(t) = 0, \quad t \in \mathcal{I}, \tag{6}$$

has local matrix pencils that are regular with index 2. Here, all vector functions given by

$$x(t) = \gamma(t) \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad t \in \mathcal{I},$$

with an arbitrary continuously differentiable scalar function γ are solutions.

In addition to the nonsingularity of $cA(t) + B(t)$ for all $t \in \mathcal{I}$, a further special structural demand is incorporated into the regularity notion in [1], which is marked as Ω-*condition*. Even though this property does no longer appear as an ingredient of the regularity notion later on, see [2], it plays a central role in Boyarintsev's work. The property Ω applies to the transformed DAE

$$\tilde{A}(t)\tilde{x}'(t) + \tilde{x}(t) = \tilde{q}(t), \quad t \in \mathcal{I}, \tag{7}$$

which is supposed to satisfy the following basic conditions (see [1, p. 73 and p. 83]): The Jordan representation $\tilde{A} = NJN^{-1}$ is valid with continuously differentiable nonsingular N, N^{-1} and continuous J. Additionally, the Drazin inverse J^D is continuous, and hence $\operatorname{ind} J(t)$ is time-invariant, $J^D J$ is constant and $J^s(I - J^D J)$ is constant for $s \in \mathbb{N}$. This implies the particular structure

$$\tilde{A}(t) = N(t)J(t)N(t)^{-1}, \quad J(t) = \begin{bmatrix} J_0 & 0 \\ 0 & J_1(t) \end{bmatrix} \tag{8}$$

with a constant nilpotent block J_0 and a nonsingular block $J_1(t)$. Since $\ker J(t)$ is time-invariant, it also follows that (cf. (2))

$$\ker A(t) = \ker \tilde{A}(t) = \ker\,(J(t)N(t)^{-1}) = N(t)\ker J(t) \tag{9}$$

is a $\mathcal{C}^1$- subspace varying with t in $\mathbb{R}^m$.

Next, for the Ω-condition, see [1, p. 90], the gist of the matter is the following:

Definition 1. *The time-varying matrix $\tilde{A}(t)$ has the property Ω on the interval $\mathcal{I}$ if at least one of the following three conditions is valid:*

1. *$\tilde{A}(t)$ is nonsingular for $t \in \mathcal{I}$,*

2. *$\tilde{A}(t)$ has index 1 for $t \in \mathcal{I}$,*

3. *$\tilde{A}(t)$ can be brought into Jordan form by a constant similarity transform, that is, $\tilde{A}(t) = NJ(t)N^{-1}$ for $t \in \mathcal{I}$.*

By the Ω-condition, if $\mathrm{ind} J_0 \geq 2$, then N is supposed to be time-invariant. In this case, the subspaces $\ker \tilde{A}(t)$ and $\mathrm{im}\, \tilde{A}(t)$ are also time-invariant, which rules out the negative Example 3. Note that Definition 3.7.2 in [2] generalizes the Ω-condition in the sense that the expression $(I - A(t)^D A(t))A(t)$ is time-invariant on the given interval.

2. Regularity in the projector based framework

As before we consider the DAE (1) with continuous coefficients and a $\mathcal{C}^1$- subspace $\ker A$, but we do not demand regular local matrix pencils.
By means of a continuously differentiable projector valued function

$$P : \mathcal{I} \to \mathbb{R}^{m,m},$$

with $\ker P(t) = \ker A(t)$, $P(t)^2 = P(t)$ for all $t \in \mathcal{I}$, we rewrite the DAE (1) as *DAE with properly stated leading term*

$$A(t)(P(t)x(t))' + (B(t) - A(t)P'(t))x(t) = q(t), \quad t \in \mathcal{I}, \tag{10}$$

and, aiming at a further analysis of the DAE, we construct a sequence of admissible matrix functions and associated projector functions. Thereby, the special choice of the projector function P does not matter at all. One can restrict oneself to deal with the orthoprojector function.
For more transparency, we drop the argument t in most parts; the relations a meant pointwise on the given interval. Below, the condition $Q_i = Q_i^2$ indicates that Q_i is a projector valued function. We adapt the notion of admissible matrix function sequences [10, Definition 1.10] to the DAE (10):

Definition 2. *The sequence of continuous matrix functions $G_0, \ldots, G_\kappa$ is said to be admissible on the interval $\mathcal{I}$ for the DAE (10) if it is built by the following rule:*

$$\begin{aligned}
&G_0 := A,\ Q_0 = Q_0^2,\ \mathrm{im}\, Q_0 = N_0 := \ker G_0,\ \varPi_0 := P_0 := I - Q_0,\\
&B_0 := B - AP',\\
&\mathit{for} \quad i = 0, \ldots, \kappa - 1:\\
&G_{i+1} := G_i + B_i Q_i,\\
&\qquad N_{i+1} := \ker G_{i+1},\ \widehat{N}_{i+1} := N_{i+1} \cap (N_0 + \cdots + N_i),\\
&\qquad Q_{i+1} = Q_{i+1}^2,\ \mathrm{im}\, Q_{i+1} = N_{i+1},\ (N_0 + \cdots + N_i) \ominus \widehat{N}_{i+1} \subseteq \ker Q_{i+1},\\
&\qquad P_{i+1} := I - Q_{i+1},\ \varPi_{i+1} := \varPi_i P_{i+1},\\
&B_{i+1} := B_i P_i - G_{i+1} P (P\varPi_{i+1})' P\varPi_i,
\end{aligned}$$

and Q_0 is continuous, $P\varPi_1, \ldots, P\varPi_\kappa$ are continuously differentiable, and G_i has constant rank r_i for $i = 0, \ldots, \kappa$.

By construction, it holds that $r_i \leq r_{i+1}$.

Definition 3. *The DAE (1) is said to be regular on the interval $\mathcal{I}$ if $r_0 = m$ or if there are an integer $\mu \in \mathbb{N}$ and an admissible matrix function sequence $G_0, \ldots, G_\mu$ constructed for (10) such that*

$$0 \leq r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m. \tag{11}$$

Put $\mu = 0$ if $r_0 = m$.
The values r_i are called characteristic values and μ is called the tractability index of the DAE (1).

This definition is consistent with [10, Definition 2.61]. We refer to [10, Chapter 2], for arguments such as solvability properties justifying the term regularity as regards content.
If the DAE is regular, then the above intersection subspaces are necessarily trivial, that is,

$$\widehat{N}_{i+1} = \{0\}, \quad \text{for } i = 0, \dots, \mu - 2.$$

Besides, this condition represents a practically useful regularity criterion.
In the context given here, we can restrict ourselves to the so-called *widely orthogonal projector functions* [10, Subsubsection 2.2.3] in the regular case, which correspond to the special choice

$$\ker Q_{i+1} = (N_0 + \cdots + N_i) \oplus (N_0 + \cdots + N_{i+1})^{\perp}, \quad i = 0, \dots \mu - 2. \tag{12}$$

The tractability index as well the characteristic values of the DAE are invariant under scalings of the DAE and transformations of the unknown function. Also refactorizations of the leading term, e.g. choosing a new projector function P in (10), do not change these values, [10]. In the case of constant coefficients A and B, these values describe the structure of the Kronecker normal form of the matrix pencil formed by the ordered pair $\{A, B\}$.

The reformulation (10) suggests to consider also the so-called *modified local matrix pencils*

$$\lambda A(t) + B(t) - A(t)P'(t), \quad t \in \mathcal{I}. \tag{13}$$

If the DAE is regular with tractability index 0, then $A(t)$ remains nonsingular on the given interval and $P(t) = I$. Then, trivially, the local matrix pencils and the modified local matrix pencils are regular, uniformly with Kronecker index 0.
If the DAE is regular with tractability index 1, then the local matrix pencils and the modified local matrix pencils are regular, uniformly with Kronecker index 1, e.g., [8, Theorem A.13].
The classes of regular DAEs with tractability index $\mu \in \{0, 1\}$ coincide in essence with those described in Definition 1, item 1 and item 2. We concentrate now on the more complicated higher index cases.

Motivated by a series of examples, it has been the conjecture in [8, 9] that the modified local pencil has stronger relevance for global DAE properties. Notice that the regular index-2 DAE (4) actually shows regular modified local pencils, while the modified local pencil of the nonregular DAE (6) is singular as expected. However, for the regular index-3 DAE (5) also the modified local pencil fails to be regular, and hence, this conjecture appears to be a misapprehension for regular DAEs with index $\mu \geq 3$, while it becomes true for regular index-2 DAEs. Note that the early version of tractability index 2 in [8] applies modified local pencils.

Theorem 1. *The DAE (1) is regular with tractability index 2 on the interval $\mathcal{I}$ if and only if the modified local matrix pencils (13) are regular, uniformly with Kronecker index 2.*
The local matrix pencils of a DAE (1) that are regular with tractability index 2 are not necessarily regular.

Proof. The first statement is a consequence of [12, Theorem 2.6], and, due to [8, Theorem 1.3.1.], the property of the modified local matrix pencils to be regular with Kronecker index 2 is independent of the choice of the projector function P.
The DAE (4) in Example 1 is regular with tractability index 2, since it represents a transformed constant coefficient index-2 DAE. The local matrix pencils are singular, which confirms the second statement. □

Corollary 1. *If the DAE (1) is regular with tractability index 2 and if, additionally, the nullspace* $\ker A(t)$ *is time-invariant, then the local matrix pencils are regular, uniformly with index 2.*

Proof. This statement follows from Theorem 1 by taking P as the orthoprojector function. □

Note that there is a variety of possibilities to choose the projector functions $Q_0, \ldots, Q_{\mu-1}$. Of course, the characteristic values do not depend on the special choice of the projector function, while the matrix functions themselves do so. We emphasize that the sum-subspaces

$$N_0,\ N_0 + N_1, \ldots, N_0 + \cdots + N_{\mu-1} \tag{14}$$

are also independent of the special choice of the projector functions, [10]. We refer to [10] for further properties. Regular time-varying DAEs are shown to be solvable similar to constant coefficient DAEs corresponding to regular matrix pencils.

Proposition 1. *If the leading matrix coefficient* $A(t)$ *in (1) has a time-invariant nullspace, then the local matrix pencils and the modified local matrix pencils coincide.*

Proof. Denote by P_c and P a constant and an arbitrary continuously differentiable projector function along $\ker A$, respectively. Then we have $AP' = AP_cP' = A(P_cP)' = A(P_c)' = 0$. □

The next theorem generalizes Corollary 1 for DAEs that are regular with arbitrary index.

Theorem 2. *Let the DAE (1) be regular with tractability index* $\mu \geqq 2$ *and characteristics (11), and let the subspaces*

$$N_0,\ N_0 + N_1, \ldots, N_0 + \cdots + N_{\mu-2} \tag{15}$$

be time-invariant.
Then the local matrix pencils are regular, with uniform Kronecker index μ *and characteristics (11).*

Proof. We apply P to be the orthoprojector onto (the time invariant) subspace $(\ker A)^{\perp}$, which yields $B_0 = B$. We choose Q_0 to be the orthoprojector onto $\ker G_0 = \ker A$ such that $P_0 = P$. Since the DAE (1) is regular, so is the alternative version (10) with properly stated leading term. In particular, we can apply the so-called widely orthogonal projector functions Q_0 and $Q_1, \ldots, Q_{\mu-1}$, see [10]. The resulting projector functions $\Pi_0, \ldots, \Pi_{\mu-1}$ are the orthoprojectors along the subspaces $N_0, \ldots, N_0 + \cdots + N_{\mu-1}$ and, hence, $\Pi_0, \ldots, \Pi_{\mu-2}$ must be time-invariant. Moreover, also $P\Pi_0 = \Pi_0, \ldots, P\Pi_{\mu-2} = \Pi_{\mu-2}$ are time-invariant. Therefore, the expressions $(P\Pi_{i+1})'$ within the matrix function sequence disappear for $i = 0, \ldots, \mu - 3$. It results that

$$B_{i+1} = B_i P_i = B\Pi_i, \quad i = 0, \ldots, \mu - 3,$$

and further, with $F := I - P_{\mu-1}P(\Pi_{\mu-1})'\Pi_{\mu-2}Q_{\mu-1}$ being nonsingular,

$$\begin{aligned} G_{\mu-1} &= A + B(Q_0 + \Pi_0 Q_1 + \cdots + \Pi_{\mu-3}Q_{\mu-2}), \\ B_{\mu-1} &= B_{\mu-2}P_{\mu-2} - G_{\mu-1}P(\Pi_{\mu-1})'\Pi_{\mu-2} = B\Pi_{\mu-2} - G_{\mu-1}P(\Pi_{\mu-1})'\Pi_{\mu-2}, \end{aligned}$$

$$\begin{aligned} G_\mu &= G_{\mu-1} + B_{\mu-1}Q_{\mu-1} = G_{\mu-1} + B\Pi_{\mu-2}Q_{\mu-1} - G_{\mu-1}P(\Pi_{\mu-1})'\Pi_{\mu-2}Q_{\mu-1} \\ &= (G_{\mu-1} + B\Pi_{\mu-2}Q_{\mu-1})F \\ &= (A + B(Q_0 + \Pi_0 Q_1 + \cdots + \Pi_{\mu-2}Q_{\mu-1})F \\ &= (A + B(I - \Pi_{\mu-1}))F. \end{aligned}$$

Now it becomes clear that, at each frozen $t \in \mathcal{I}$, we obtain a matrix sequence for the local pencil formed by the ordered pair $\{A(t), B(t)\}$. Due to [9, Theorems 3 and 4], this pencil is regular with Kronecker index μ and characteristics (11).

$\square$

Turn back to the DAE class considered by Boyarintsev, given by (7), (8) and Definition 1 (3), i.e.,

$$N \begin{bmatrix} J_0 & 0 \\ 0 & J_1(t) \end{bmatrix} N^{-1}\tilde{x}'(t) + \tilde{x}(t) = \tilde{q}(t), \quad t \in \mathcal{I}. \tag{16}$$

Let the constant matrix J_0 have index $\mu \geq 2$, and let $G_0^{J_0} := J_0$, $G_1^{J_0}, \dots, G_\mu^{J_0}$ denote an admissible matrix sequence for the matrix pair $\{J_0, I\}$ associated with the projectors $Q_0^{J_0}, \dots, Q_{\mu-1}^{J_0}$. All matrices $G_i^{J_0}$ and $Q_i^{J_0}$ are constant, and so are the subspaces $N_i^{J_0} := \ker G_i^{J_0}$ for $i \leq \mu$. Moreover, $G_i^{J_0}$ is singular for $i \leq \mu - 1$, but $G_\mu^{J_0}$ is nonsingular.
With

$$\begin{aligned} \tilde{G}_i(t) &:= N \begin{bmatrix} G_i^{J_0} & 0 \\ 0 & J_1(t) \end{bmatrix} N^{-1}, \tilde{Q}_i(t) := N \begin{bmatrix} Q_i^{J_0} & 0 \\ 0 & I \end{bmatrix} N^{-1}, \quad i = 0, \dots, \mu - 1, \\ \tilde{G}_\mu(t) &:= N \begin{bmatrix} G_\mu^{J_0} & 0 \\ 0 & J_1(t) \end{bmatrix} N^{-1}, \quad t \in \mathcal{I}, \end{aligned}$$

we obtain an admissible matrix function sequence and associated time-invariant projector functions for the DAE (16). The corresponding subspaces

$$\tilde{N}_0 = N(N_0^{J_0} \times \{0\}), \dots, \tilde{N}_{\mu-1} = N(N_{\mu-1}^{J_0} \times \{0\}),$$

are time-invariant and, hence, $\tilde{N}_0$, $\tilde{N}_0 + \tilde{N}_1$, up to $\tilde{N}_0 + \cdots + \tilde{N}_{\mu-1}$ are so. This shows that Theorem 2 applies to the DAE (16). Taking into account that the DAE (16) represents the special form of the DAE (2) corresponding to Boyarintsev's Ω-condition, and regarding that the scaling of the DAE by $(cA(t) + B(t))^{-1}$ and the transformation $x(t) = K(t)\tilde{x}(t)$, $K(t) := \exp(ct)I$, do not change these sum-subspaces, we know that Boyarintsev's higher index DAEs have time-invariant sum-subspaces (14) in their original form (1). This proves that Boyarintsev's class of higher index DAEs represents a special class belonging to the application field of Theorem 2.

3. Checking time-invariant sum-subspaces and the responsibility of local pencils

Recall once again that the local matrix pencils of index-0 and index-1 DAEs are always regular with index 0 and index 1, respectively.

For regular higher-index DAEs, Theorem 2 provides a useful, *sufficient criterion* of the responsibility for the local matrix pencils. The subspaces (15) are time-invariant if and only if the projector functions $\Pi_0, \dots, \Pi_{\mu-2}$ associated with widely orthogonal nullspace- projectors $Q_0, \dots, Q_{\mu-2}$ are so. In consequence, having no further a priori information, one has to generate the projector functions $\Pi_0, \dots, \Pi_{\mu-2}$ associated with widely orthogonal nullspace-projectors $Q_0, \dots, Q_{\mu-2}$ by means of one of the algorithms described in [10, Chapter 7] first, and then to check their time-invariance. The latter can be done by applying a difference. For low-dimension problems, algorithmic differentiation techniques (AD) are preferable. AD provides the Taylor coefficients of Π_i and, therefore, a more reliable check of time-invariance, see [11], [5].

Of course, if available, an a priori structural analysis ensuring structure reflecting local matrix pencils and even time-invariant subspaces (15) would be best. At this place, we emphasize once again that time-invariant subspaces (15) represent a sufficient, but not a necessary condition for the structure preservation of the local pencils. For instance, the DAEs in Hessenberg form show regular local matrix pencils reflecting exactly the given Hessenberg form though the subspaces (15) may vary with time. Take a look at the special case of a DAE in size-2 Hessenberg form:

$$\begin{aligned} x_1'(t) + B_{11}(t)x_1(t) + B_{12}(t)x_2(t) &= q_1(t), \\ B_{21}(t)x_1(t) &= q_2(t), \end{aligned}$$

comprising $m_1 + m_2 = m$ equations, with $B_{12}(t)B_{21}(t)$ being nonsingular everywhere. This DAE is regular with tractability index 2. Its characteristics are $r_0 = r_1 = m_1$, $r_2 = m$. We have further

$$G_0(t) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, Q_0(t) = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, G_1(t) = \begin{bmatrix} I & B_{12}(t) \\ 0 & 0 \end{bmatrix},$$

and

$$\begin{aligned} N_0(t) &= \{z \in \mathbb{R}^{m_1+m_2} : z_1 = 0\}, \\ N_1(t) &= \{z \in \mathbb{R}^{m_1+m_2} : z_1 + B_{12}(t)z_2 = 0\} \\ &= \{z \in \mathbb{R}^{m_1+m_2} : z_1 \in \operatorname{im} B_{12}(t), B_{12}(t)^- z_1 + z_2 = 0\}, \\ N_0(t) + N_1(t) &= \operatorname{im} B_{12}(t) \times \mathbb{R}^{m_2}. \end{aligned}$$

If $\operatorname{im} B_{12}$ varies with time, then so does the subspace $N_0(t) + N_1(t)$. Then, Theorem 2 does not apply and also the Ω-condition is not given. Nevertheless, the local matrix pencils are regular with Kronecker index 2, and this remains unaffected if $\operatorname{im} B_{12}$ actually varies or not.

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ДИФФЕРЕНЦИАЛЬНО-АЛГЕБРАИЧЕСКИЕ УРАВНЕНИЯ С РЕГУЛЯРНЫМИ ЛОКАЛЬНЫМИ ПУЧКАМИ МАТРИЦ

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В рамках проекторного анализа, каждое регулярное линейное ДАУ включает в себя несколько непрерывных изменяющихся во времени характеристических подпространств, не зависящих от технической конструкции, в том числе так называемую сумму подпространств. Как известно, локальные пучки матриц линейного ДАУ, изменяющегося во времени, более высокого индекса, не отражают глобальной структуры ДАУ вообще. Покажем, что на заданном интервале, локальные пучки матриц из ДАУ регулярны и отражают глобальную структуру ДАУ если некоторые из этих характеристических подпространств стационарны. Мы обсуждаем практические методы проверки стационарности этих подпространств. Соответствующий класс ДАУ связан с классом ДАУ, ранее введенных и исследованных Ю.Е. Бояринцевым.

Ключевые слова: изменяющиеся во времени ДАУ; локальный пучок матриц; регулярность.

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