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ON ONE SOBOLEV TYPE MATHEMATICAL MODEL IN QUASI-BANACH SPACES

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The theory of Sobolev type equations experiences an epoch of blossoming. In this article the theory of higher order Sobolev type equations with relatively spectrally bounded operator pencils, previously developed in Banach spaces, is transferred to quasi-Banach spaces. We use already well proved for solving Sobolev type equations phase space method, consisting in reduction of singular equation to regular one defined on some subspace of initial space. The propagators and the phase space of complete higher order Sobolev type equations are constructed. Abstract results are illustrated by specific examples. The Boussinesq-Love equation in quasi-Banach space is considered as application.

Keywords: Sobolev type equations; quasi-Banach spaces; propagators; phase space.

Introduction. Let $\mathfrak{U}, \mathfrak{F}$ be Banach spaces, operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ (i.e. linear, continuous, defined on $\mathfrak{U}$ and acting into $\mathfrak{F}$). Consider a Sobolev type equation (the term was introduced by R.E. Showalter [1])

$$Au'' = B_1 u' + B_0 u, \ \ker A \neq \{0\}. \tag{1}$$

Sobolev type equations have been already well researched. The first monograph devoted to such equations was published in 2003 [2]. Here degenerate analytical (semi)group, and degenerate C_0-semigroups used in the study of equations of the first order were constructed. Linear Sobolev type equations of higher order in Banach spaces were studied in [3]. The results of Sobolev type equations theory are used in the theory of dynamical measurements [4], optimal control theory [5, 6], in the study of dichotomies of equations of the form (1) [7, 8]. In addition, the theory of degenerate groups and semigroups of operators was transferred into a locally convex spaces.

Equations that are not solved with respect to the highest derivative in time were studied for the first time by A. Poincare, however, a systematic study of them was started in the middle of the last century after the fundamental work of S.L. Sobolev. Now the Sobolev type equations theory is actively studied area of nonclassical equations of mathematical physics, and a number of monographs, completely devoted to them, or in part, is growing like an avalanche.

In this paper Sobolev type equations of the second order of the form (1) are considered in quasi-Banach spaces. As it is well known [9, p. 3.10], a quasi-Banach space is not a normed one, but it can be made metrizable. One of the examples of a quasi-Banach space is a space of sequences ℓ_q, $q \in (0,1)$. In [10] there was constructed a quasi-Banach space ℓ_q^m, $q \in (0,1)$, $m \in \mathbb{R}$, $\ell_q^0 = \ell_q$ which was called a quasi-Sobolev space. These spaces we will be used to illustrate the abstract results of the paper.

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1. Relatively Bounded Operator Pencils in Quasi-Banach Spaces. A lineal $\mathfrak{U}$ over a field $\mathbb{R}$ is called a *quasi-normed*, if there is a function ${}_{\mathfrak{U}}\|\cdot\| : \mathfrak{U} \to \mathbb{R}$ with the following properties:

(i) ${}_{\mathfrak{U}}\|u\| \geq 0$ for all $u \in \mathfrak{U}$, and ${}_{\mathfrak{U}}\|u\| = 0$ exactly when $u = \mathbf{0}$, where $\mathbf{0}$ is a zero in lineal $\mathfrak{U}$;

(ii) ${}_{\mathfrak{U}}\|\alpha u\| = |\alpha|{}_{\mathfrak{U}}\|u\|$ for all $u \in \mathfrak{U}$, $\alpha \in \mathbb{R}$;
(iii) ${}_{\mathfrak{U}}\|u+v\| \leq C({}_{\mathfrak{U}}\|u\| +_{\mathfrak{U}} \|v\|)$ for all $u, v \in \mathfrak{U}$, where the constant $C \geq 1$.

A function ${}_{\mathfrak{U}}\| \cdot \|$ with properties (i) – (iii) is called *a quasi-norm*. In particular, if $C = 1$ a quasi-norm ${}_{\mathfrak{U}}\| \cdot \|$ is called a norm, and lineal $\mathfrak{U}$ with the norm ${}_{\mathfrak{U}}\| \cdot \|$ is a normed one. A quasi-normed lineal $(\mathfrak{U}; {}_{\mathfrak{U}}\| \cdot \|)$ is metrizable (see [9, Lemma 3.10.1]), so we have a conception of the fundamental (or Cauchy) sequence $\{u_k\} \subset \mathfrak{U}$: ${}_{\mathfrak{U}}\|u_k - u_l\| \to 0$ for $k, l \to \infty$. Define the *quasi-Banach space* as a full quasi-normed lineal.

Example 1. Let $\{\lambda_k\} \subset \mathbb{R}_+$ be a monotonic sequence such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$, and $q \in \mathbb{R}_+$. Put

$$\ell_q^m = \left\{ u = \{u_k\} \subset \mathbb{R} : \sum_{k=1}^{\infty} \left(\lambda_k^{\frac{m}{2}} |u_k| \right)^q < +\infty \right\}.$$

Lineal ℓ_q^m for all $m \in \mathbb{R}$, $q \in \mathbb{R}_+$ with a quasi-norm of element $u = \{u_k\} \in \ell_q^m$

$${}_q^m\|u\| = \left(\sum_{k=1}^{\infty} \left(\lambda_k^{\frac{m}{2}} |u_k| \right)^q \right)^{1/q}$$

is a quasi-Banach space (for $q \in [1, +\infty)$ it is a Banach space). Note that if $q \in (0, 1)$, then constant $C = 2^{1/q}$ in (iii). The spaces ℓ_q^m are called *quasi-Sobolev* in [10].

Let $\mathfrak{U}; {}_{\mathfrak{U}}\| \cdot \|)$ and $(\mathfrak{F}; {}_{\mathfrak{F}}\| \cdot \|)$ be quasi-Banach spaces, a linear operator $L : \mathfrak{U} \to \mathfrak{F}$ with the domain $\mathrm{dom} L = \mathfrak{U}$ is called *continuous* if $\lim\limits_{k\to\infty} Lu_k = L\left(\lim\limits_{k\to\infty} u_k\right)$ for any the sequence $\{u_k\} \subset \mathfrak{U}$ which is convergent in $\mathfrak{U}$. Note that in this case the linear operator $L : \mathfrak{U} \to \mathfrak{F}$ is continuous precisely when it is bounded (i.e., displays bonded sets to bounded sets). Denote by $\mathcal{L}(\mathfrak{U}; \mathfrak{F})$ a lineal (over the field $\mathbb{R}$) of linear bounded operators. It is a quasi-Banach space with quasi-norm

$${}_{\mathcal{L}(\mathfrak{U};\mathfrak{F})}\|L\| = \sup_{{}_{\mathfrak{U}}\|u\|=1} {}_{\mathfrak{F}}\|Lu\|.$$

Now let the operators $A, B_1, B_0 \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. Following [3], introduce the sets $\rho^A(\vec{B}) = \{\mu \in C : (\mu^2 A - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and $\sigma^A(\vec{B}) = C \backslash \rho^A(\vec{B})$, which are called an *A-resolvent set* and an *A-spectrum* of the pencil $\vec{B}$, respectively.

The operator-function $R_\mu^A(\vec{B}) = (\mu^2 A - \mu B_1 - B_0)^{-1}$ with domain $\rho^A(\vec{B})$ is called an *A-resolvent* of pencil $\vec{B}$.

Definition 1. *A pencil of operators $\vec{B}$ is called polynomially relatively bounded with respect to operator A (or polynomially A-bounded) if*

$$\exists a \in R_+ \; \forall \mu \in C \; (|\mu| > a) \Rightarrow (R_\mu^A(\vec{B}) \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})).$$

Fix $\gamma = \{\mu \in C : /\mu| = r > a\}$ which is a contour bounding a disk containing $\sigma^A(\vec{B})$. We require additional condition

$$\int_\gamma R_M^A(\vec{B}) d\mu = 0. \tag{A}$$

This condition was introduced in [3] and is very important in considering of the Sobolev type equations of higher order. Note that if there exists an operator $A^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ or operator $B_1 = \mathbb{O}$ (equation is incomplete), then condition (A) holds; and if operator $A = \mathbb{O}$ and there exists an operator $B_1^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$, then condition (A) is not fulfilled.

Let the pencil $\vec{B}$ be polynomially A bounded, and (A) be fulfilled. Then there exist the following integrals of analytic operator functions:

$$P = \frac{1}{2\pi i}\int_\gamma \mu R^A_\mu(\vec{B})Ad\mu,\ Q = \frac{1}{2\pi i}\int_\gamma \mu A R^A_\mu(\vec{B})d\mu.$$

Lemma 1. *Let the pencil $\vec{B}$ be polynomially A-bounded and (A) be fulfilled.Then the operators $P \in \mathcal{L}(\mathfrak{U})$ and $Q \in \mathcal{L}(\mathfrak{F})$ are projectors.*

Denote $\mathfrak{U}^0 = \ker P,\ \mathfrak{F}^0 = \ker Q,\ \mathfrak{U}^1 = \mathrm{im}\ P,\ \mathfrak{F}^1 = \mathrm{im}\ Q$. By lemma $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1, \mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. By A^k (B^k_l) denote restriction of operator A (B_l) onto $\mathfrak{U}^k$, $k, l = 0, 1$.

Theorem 1. *Let the pencil $\vec{B}$ be polynomially A bounded, and (A) be fulfilled. Then actions of operators split:*

(i) $A^k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k),\ k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k),\ k, l = 0, 1$;
(iii) there exists an operator $(A^1)^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$;
(iv) there exists an operator $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$.

Example 2. Using formula $\Lambda u = \{\lambda_k u_k\},\ u \in \ell^m_q$ we introduce *a Laplace quasi-operator*. It is easy to show [10] that operator $\Lambda : \ell^{m+2}_q \to \ell^m_q$ is a toplinear isomorphism for all $m \in \mathbb{R}$, $q \in \mathbb{R}_+$. The inverse operator $\Lambda^{-1}u = \{\lambda^{-1}_k u_k\}$ is called *a Green quasi-operator.*

Next, construct the operators $A = \lambda - \Lambda$ and $B_1 = \alpha(\Lambda - \lambda')$, $B_0 = \beta(\Lambda - \lambda")$ $\alpha, \beta \in \mathbb{R}_+$.

Lemma 2. *Let*

(i) $\lambda \notin \{\lambda_k\}$ or (ii) $(\lambda \in \{\lambda_k\}) \wedge (\lambda \neq \lambda')$. Then the pencil $\vec{B} = (B_1, B_0)$ is polynomially A-bounded, moreover, ∞ is a removable singularity of A-resolvent of $\vec{B}$.

(iii) $(\lambda \in \{\lambda_k\}) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda")$. Then the pencil $\vec{B} = (B_1, B_0)$ is polynomially A-bounded moreover, ∞ is a pole of order 1 of A-resolvent of $\vec{B}$.

Remark 1. In case (i) A-spectrum of pencil $\vec{B}$ $\sigma^A(\vec{B}) = \{\mu^{1,2}_k : k \in \mathbb{N}\}$, where $\mu^{1,2}_k$ are the roots of equation

$$(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda\lambda_k) = 0. \tag{2}$$

In case (ii) $\sigma^A(\vec{B}) = \{\mu^{1,2}_k : k \in \mathbb{N},\ \lambda_k \neq \lambda\} \cup \{\mu_l : l \in \mathbb{N},\ \lambda_l = \lambda\}$, where $\mu^{1,2}_k$ are the roots of (2) when $\lambda = \lambda_k$, and μ_l is the root of (2) when $\lambda = \lambda_l$. In case (iii) $\sigma^A(\vec{B}) = \{\mu^{1,2}_k : k \in \mathbb{N},\ \lambda_k \neq \lambda\}$.

Remark 2. It is easy to see that if $(\lambda \in \{\lambda_k\}) \wedge (\lambda = \lambda' = \lambda")$ then the pencil $\vec{B}$ is not polynomially A-bounded.

Now let's check the condition (A). In case (i) there exists an operator $A^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$, so that condition (A) holds. In case (ii)

$$\frac{1}{2\pi i}\int\limits_\gamma \sum_{k=1}^{\infty} \frac{<\varphi_k, \cdot> \varphi_k d\mu}{(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda\lambda_k)} = \sum_{\lambda=\lambda_k} \frac{<\varphi_k, \cdot>}{\alpha(\lambda' - \lambda_k)}\varphi_k \neq 0,$$

i.e. (A) is not fulfilled, so this case is excluded from further considerations. In case (iii) (A) holds.

2. The Phase Space and Propagators. Now consider the Sobolev type equation of the second order (1) with the initial condition

$$u^{(m)}(t) = u(m, \ m = 0, 1. \tag{3}$$

Definition 2. *The operator-function* $\mathfrak{U}^{\bullet} \in C^{\infty}(\mathbb{R}; \mathcal{L}(\mathfrak{U}))$ *is called a propagator of equation* (1) *if for any* $u \in \mathfrak{U}$ *vector- function* $u(t) = U^t u$ *is a solution of this equation.*

Lemma 3. *Let the pencil* $\vec{B}$ *be polynomially* A*-bounded and condition* (A) *be fulfilled. Then there exist propagators of equation* (1)*:*

$$U_1^t = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^A(\vec{B}) A e^{\mu t} d\mu, t \in \mathbb{R}, \quad U_0^t = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^A(\vec{B})(\mu(A - B_1) e^{\mu t} d\mu, t \in \mathbb{R}.$$

Further, if the pencil $\vec{B}$ is polynomially A-bounded, condition (A) holds and ∞ is a pole of order $p \in \{0\} \bigcup \mathbb{N}$ of A-resolvent of $\vec{B}$, then the pencil $\vec{B}$is called (A, p)-bounded.

Definition 3. *The Set* $\mathfrak{P} \subset \mathfrak{U}$ *is called a phase space of equation* (1) *if*
(i) for any $u_j \in \mathfrak{P}, \ j = 0, 1,$ *there exists a unique solution of* (1), (3)*;*
(ii) any solution $u = u(t)$ *of equation* (1) *lies in* $\mathfrak{P}$ *as a trajectory (i.e.,* $u(t) \in \mathfrak{P}$ *for all* $t \in \mathbb{R}$*).*

Theorem 2. *Let the operator* M *be* (L, p)*-bounded,* $p \in \{0\} \cup \mathbb{N}$*. Then the subspace* $\mathfrak{U}^1$ *is a phase space of equation* (1)*. Moreover, for any* $u_j \in \mathfrak{U}^1, \ j = 0, 1,$ *there exists a unique solution of the Cauchy problem* (3) *for equation* (1) *which can be represented as:* $u(t) = U_0^t u_0 + U_1^t u_1$*.*

Example 3. Let $\mathfrak{U} = \ell_q^{m+2}$, $\mathfrak{F} = \ell_q^m$, $m \in \mathbb{R}$, $q \in (0, 1)$, where ℓ_q^m is a quasi-Sobolev space defined in example 1, and A, B_1, B_0 are the operators constructed in example 2.

Consider the Boussinesq–Love equation as one of the most well-known non-classical equations of mathematical physics of the second order in time [11]

$$(\lambda - \Lambda)\ddot{u} = \alpha(\Lambda - \lambda')\dot{u} + \beta(\Lambda - \lambda")u, \ u(t) \in \mathfrak{U}. \tag{4}$$

Theorem 3. *Let* $\lambda \notin \{\lambda_k\}$ *or* $(\lambda \in \{\lambda_k\}) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda")$*. Then for any sequence*

$$u_j = \{u(jk\} \in \mathfrak{U}^1 = \begin{cases} \mathfrak{U}, \text{ if } \lambda \neq \lambda_k, \ k \in \mathbb{N}; \\ \{u \in \mathfrak{U} : \ u_l = 0, \ \lambda = \lambda_l\}, \end{cases} \ j = 0, 1$$

there exists a unique solution of (3), (4)*, which also has the form*

$$u(t) = \sum {}' \left[\frac{\mu_k^1(\lambda - \lambda_k) + \alpha(\lambda' - \lambda_k)}{(\lambda - \lambda_k)(\mu_k^1 - \mu_k^2)} e^{\mu_k^1 t} + \frac{\mu_k^2(\lambda - \lambda_k) + \alpha(\lambda' - \lambda_k)}{(\lambda - \lambda_k)(\mu_k^2 - \mu_k^1)} e^{\mu_k^2 t} \right] u_{0k} e_k +$$

$$+ \sum {}' \frac{e^{\mu_k^1 t} - e^{\mu_k^2 t}}{(\mu_k^1 - \mu_k^2)} u_{1k} e_k,$$

where the single quote at the sum means the absence of terms with numbers k *such that* $\lambda = \lambda_k$*. Here* $\mu_k^{1,2}$ *are the roots of* (2)*, the vectors* $e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$*, where unit stands on the* k*-th place. The set* $\mathfrak{U}^1$ *is a phase space of equation* (4)*.*

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ОБ ОДНОЙ МАТЕМАТИЧЕСКОЙ МОДЕЛИ СОБОЛЕВСКОГО ТИПА В КВАЗИБАНАХОВЫХ ПРОСТРАНСТВАХ

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Теория уравнений соболевского типа переживает эпоху бурного расцвета. В данной работе теория уравнений соболевского типа высокого порядка с относительно спектрально ограниченным пучком операторов, развитая в банаховых пространствах, переносится в квазибанаховы пространства. Мы используем уже хорошо зарекомендовавший себя при решении уравнений соболевского типа метод фазового пространства, заключающийся в редукции сингулярного уравнения к регулярному, определенному на некотором подпространстве исходного пространства. Построены пропагаторы и фазовое пространство полного уравнения соболевского типа второго порядка. Абстрактные результаты иллюстрированы конкретными примерами. В качестве приложения рассмотрено уравнение Буссинеска – Лява в квазибанаховом пространстве.

Ключевые слова: уравнения соболевского типа высокого порядка; квазибанаховы пространства; пропагаторы; фазовое пространство.

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