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ELLIPTIC PROBLEMS WITH ROBIN BOUNDARY COEFFICIENT-OPERATOR CONDITIONS IN GENERAL L_p SOBOLEV SPACES AND APPLICATIONS

M. Cheggag, Polytechnic National School of Oran, Oran, Algeria, mustapha.cheggag@enp-oran.dz***,***
A. Favini, University of Bologna, Bologna, Italy, favini@dm.unibo.it***,***
R. Labbas, University of Le Havre, Le Havre, France, rabah.labbas@univ-lehavre.fr***,***
S. Maingot, University of Le Havre, Le Havre, France, stephane.maingot@univ-lehavre.fr***,***
A. Medeghri, University of Mostaganem, Mostaganem, Algeria, medeghri@univ-mosta.dz

In this paper we prove some new results on complete operational second order differential equations of elliptic type with coefficient-operator conditions, in the framework of the space $L^p(0,1;X)$ with general $p \in (1,+\infty)$, X being a UMD Banach space. Existence, uniqueness and optimal regularity of the classical solution are proved. This paper improves and completes naturally our last two works on this problematic.

Keywords: second-order abstract elliptic differential equations; Robin boundary conditions; analytic semigroup.

To the memory of Alfredo Lorenzi.

Introduction and Hypotheses

In this work we study the following operational second order complete elliptic differential Problem

$$\begin{cases} u''(x) + 2Bu'(x) + Au(x) = f(x), \quad \text{a.e. } x \in (0,1), \\ u'(0) - Hu(0) = d_0, \ u(1) = u_1, \end{cases} \tag{1}$$

where A, B, H are closed linear operators in X (X being a complex Banach space), d_0, u_1 are given elements in X and $f \in L^p(0,1;X)$, $1 < p < \infty$.

We seek for a classical solution u to (1), i.e. a function u such that:

$$\begin{cases} i)\ u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)), \ u' \in L^p(0,1;D(B)), \\ ii)\ u(0) \in D(H), \\ iii)\ u \text{ satisfies } (1). \end{cases} \tag{2}$$

This optimal L^p-regularity is very important to solve many quasilinear parabolic equations corresponding to (1). In fact, the use of the fixed point theorem to solve these nonlinear problems requires necessarily optimal regularities such (2).

The Robin boundary condition $u'(0) - Hu(0) = d_0$ arises in many concrete situations and generalizes, for instance, the well known impedance boundary condition in 0. In fact we will see in the applications that we can take $H = -A$ or some fractional power of $-A$.

We first study the Problem

$$\left\{ \begin{array}{l} u''(x) + (L - M)u'(x) - LMu(x) = f(x), \quad \text{a.e. } x \in (0,1), \\ u'(0) - Hu(0) = d_0, \ u(1) = u_1, \end{array} \right. \tag{3}$$

where L and M are some closed linear operators in X. Then, in order to solve (1), we solve (3) with L and M satisfying moreover

$$L - M \subset 2B \text{ and } LM \subset -A. \tag{4}$$

Here, when P, Q are two linear operators in X, $P \subset Q$, means that $D(P) \subset D(Q)$ and $P = Q$ on $D(P)$.

By classical solution u to (3), one means that

$$\left\{ \begin{array}{l} i) \ u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(LM)), u' \in L^p(0,1;D(L-M)), \\ ii) \ u(0) \in D(H), \\ iii) \ u \text{ satisfies } (3). \end{array} \right. \tag{5}$$

Note that in virtue of (4), a classical solution u to (3) will be, a fortiori, a classical solution u to (1).

In order to solve Problems (1) and (3) for any $f \in L^p(0,1;X), 1 < p < \infty$, we will assume in all this paper that

$$X \text{ is a UMD space.} \tag{6}$$

We recall that a Banach space X is a UMD space if and only if for some $p > 1$ (and thus for all p) the Hilbert transform is continuous from $L^p(\mathbb{R};X)$ into itself (see [1, 2]).

Many authors have studied the equation

$$u''(x) + 2Bu'(x) + Au(x) = f(x), \quad \text{a.e. } x \in (0,1),$$

with the Dirichlet boundary conditions $u(0) = u_0$, $u(1) = u_1$. When $f \in L^p(0,1;X)$, $1 < p < +\infty$ (see for example [3, 4]); when $f \in C^\theta([0,1];X)$, $0 < \theta < 1$ (see [5–9]).

Here we deal with the following operational Robin boundary condition in 0, $u'(0) - Hu(0) = d_0$, which contains a general linear closed operator H. Therefore the situation is more complicated because of the different domains for instance. In the particular case $B = 0$, Problem (1) has been considered in [11] when $f \in L^p(0,1;X)$, $1 < p < +\infty$ and in [10] for $f \in C^\theta([0,1];X)$, $0 < \theta < 1$. We recall also the study [12] where B is supposed generating a group.

In this paper we will consider more general situations (see Subsection 2.1). Our techniques are based upon the Dore – Venni Theorem [13], on the sum of two closed linear operators, on the results in Prüss – Sohr paper [14] and on the reiteration Theorem in the interpolation theory, (see [15, 16]).

Let us mention that all the papers quoted above and also our study, deal with the commutative case (A, B or L, M commute in some sense).

A recent paper [17] treats one interesting non-commutative framework, for the boundary Dirichlet following problem

$$\left\{ \begin{array}{l} u''(x) + 2Bu'(x) + Au(x) - \omega u(x) = f(x), \quad \text{a.e. } x \in (0,1), \\ u(0) = u_0, \ u(1) = u_1, \end{array} \right.$$

(with $\omega > 0$ large enough). This new approach will lead us to develop a future work, solving the same equation with operational Robin boundary conditions in non-commutative situations.

The plan of the paper is as follows.

Section 2 is devoted to Problem (3); we first give our assumptions on operators L, M and H, we then give a representation formula of the solution and conclude by analyzing this representation.

In Section 3 we apply the results of section 2 with

$$L = B - (B^2 - A)^{1/2} \text{ and } M = -B - (B^2 - A)^{1/2},$$

to solve Problem (1). We also specify the assumptions on A and B.

In section 4 we study some interesting particular situations in which our assumptions on L, M and H are satisfied.

Finally in section 5 we give some concrete examples of partial differential equations to which our theory applies.

1. Study of Problem (3)

1.1. Preliminaries

First, define the class BIP(X,α) where $\alpha \in [0,\pi)$ (see [14, p. 430]): $U \in$BIP(X,α) if U is a closed linear densely defined operator satisfying

$$\left\{ \begin{array}{l} (-\infty,0) \subset \rho(U),\ N(U) = \{0\},\ \overline{R(U)} = X \\ \text{and } \exists C \geqslant 1 : \forall \lambda > 0, \left\|(U+\lambda I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant C/\lambda, \end{array} \right. \tag{7}$$

($N(U)$, $R(U)$ and $\rho(U)$ are respectively the kernel, the range and the resolvent set of U) and

$$\left\{ \begin{array}{l} \text{For all } s \in \mathbb{R}, U^{is} \in \mathcal{L}(X) \text{ and} \\ \exists C \geqslant 1 : \forall s \in \mathbb{R}, \|U^{is}\|_{\mathcal{L}(X)} \leqslant Ce^{\alpha|s|}. \end{array} \right. \tag{8}$$

We recall that operator verifying (7) admits a complex power U^z for any $z \in \mathbb{C}$ (see [18, p. 70]).

On the other hand, let $\theta \in (0,1), q \in [1,+\infty], m \in \mathbb{N}, \mu \in \mathbb{R}$ and V a closed linear operator in X satisfying

$$]\mu,+\infty) \subset \rho(V) \text{ and } \sup_{\lambda>\mu} \left\|\lambda\,(V-\lambda I)^{-1}\right\|_{\mathcal{L}(X)} < +\infty.$$

Then we consider the interpolation space $(X, D(V))_{\theta,q}$ and define

$$(X, D(V))_{m+\theta,q} := \left\{\phi \in D(V^m) : V^m\phi \in (X, D(V))_{\theta,q}\right\}.$$

When $\theta \neq 1/2$, we can use the well known following reiteration result

$$\left(X, D(V^2)\right)_{\theta,q} = (X, D(V))_{2\theta,q}\,. \tag{9}$$

We recall moreover that $(D(V), X)_{\theta,q} = (X, D(V))_{1-\theta,q}$, so using (9) we get

$$\left(D(V^2), X\right)_{\theta,q} = \left(X, D(V^2)\right)_{1-\theta,q} = (X, D(V))_{2-2\theta,q}\,, \tag{10}$$

(for details on interpolation spaces and reiteration see for instance [19]).

1.2. Hypotheses on Operators

Our assumptions on L, M and H are the following. L, M and H are linear closed operators such that

$$D(L) = D(M) \text{ and } D(ML) = D(LM), \tag{11}$$

$$ML = LM, \tag{12}$$

$$\exists \theta_L, \theta_M \in]0, \pi/2[: -L \in \text{BIP}\,(X, \theta_L) \text{ and } -M \in \text{BIP}\,(X, \theta_M)\,, \tag{13}$$

$$L + M \text{ is boundedly invertible,} \tag{14}$$

$$\begin{cases} \forall\, \xi \in D(H), \forall \lambda \in \rho(L),\ (L - \lambda I)^{-1} \xi \ \in D(H) \text{ and} \\ (L - \lambda I)^{-1} H\xi = H\,(L - \lambda I)^{-1}\xi, \end{cases} \tag{15}$$

and

$$\begin{cases} \forall \xi \in D(H), \forall \mu \in \rho(M),\ (M - \mu I)^{-1} \xi \in D(H) \text{ and} \\ (M - \mu I)^{-1} H\xi = H\,(M - \mu I)^{-1}\xi. \end{cases} \tag{16}$$

The previous assumptions allow us to build $e^{L+M} \in \mathcal{L}\,(X)$ (see Lemma 1 below) and then we can consider the linear operator Λ defined by

$$D(\Lambda) = D(L) \cap D(H) \text{ and } \Lambda = (M - H) + e^{L+M}\,(L + H)\,.$$

We will suppose that

$$\Lambda \text{ is closed and boundedly invertible.} \tag{17}$$

This last assumption signifies exactly that the determinant, in some sense, of (3) is invertible. It generalizes the hypothesis (16) (used in the paper [12, p. 526]), since when $B = 0$ they coincide. This will be discussed further (see section 3).

Our main result in this work affirms that under the above assumptions on L, M, and if $f \in L^p(0, 1; X)$ with $1 < p < \infty$, then Problem (3) has a unique classical solution u in the sense of (5) if and only if

$$\Lambda^{-1} d_0, u_1 \in (D(LM), X)_{1/2p,p}\,.$$

1.3. Consequences of Assumptions

Remark 1. Assume (11) and (12). Then

1. $D(L^2) = D(M^2) = D(ML) = D(LM)$.
2. for $\theta \in (0, 1), q \in [1, +\infty)$ we have $(X, D(L))_{\theta,q} = (X, D(M))_{\theta,q}$ and

   $$\left(X, D(L^2)\right)_{\theta,q} = \left(X, D(M^2)\right)_{\theta,q} = (X, D(LM))_{\theta,q} = (X, D(ML))_{\theta,q}\,,$$

   and also

   $$(X, D(L))_{1+\theta,q} = (X, D(M))_{1+\theta,q}\,,$$

   (for the last equality see [17, Remark 5, p. 4970]).

3. $\forall\lambda \in \rho(L), \forall\mu \in \rho(M) : (L - \lambda I)^{-1}(M - \mu I)^{-1} = (M - \mu I)^{-1}(L - \lambda I)^{-1}$.

Remark 2. Due to Prüss – Sohr [14, Theorem 2, p. 437], assumption (13) implies that L and M generate uniformly bounded analytic semigroups in X : $\left(e^{xL}\right)_{x\geqslant 0}$, $\left(e^{xM}\right)_{x\geqslant 0}$.

We then detail some properties of the sum $L + M$ and the product LM.

Remark 3. Under (6) and (11)$\sim$(13), we can apply Theorem 4, Theorem 5 and Corollary 3 in [14, p. 441, p. 443 and p. 444]. We obtain the following important results.

1. Operator $-L - M$ with domain $D(L) = D(M)$ is closed and satisfies (7). Moreover, if L or M is boundedly invertible then $L + M$ is boundedly invertible and in this case (14) is satisfied.

2. We can choose $\varepsilon > 0$ (arbitrary small) such that

$$-(L + M) \in \mathrm{BIP}(X, \theta) \text{ with } \theta = \max(\theta_L, \theta_M) + \varepsilon, \tag{18}$$

(if $\theta_L \neq \theta_M$, it is even possible to take $\varepsilon = 0$). It follows that $L + M$ generates a uniformly bounded analytic semigroup in X, but this can be proved in a another way, without applying the BIP operator theory see Lemma 1, statement 6.

3. LM is closable and $\overline{LM}$ belongs to $\mathrm{BIP}(X, \theta_L + \theta_M)$. The closability is obtained by a direct application of Corollary 3 in [14], but here, due to the fact that $D(L) = D(M)$, we can apply [4, Lemma 1, p. 168], to show that LM is closed, so $LM \in \mathrm{BIP}(X, \theta_L + \theta_M)$.

We now study some commutativity properties.

Lemma 1. *Assume (6) and (11)$\sim$(17).*

1. *Let $C \in \{M, L, L + M\}$, $\widetilde{C} \in \{M, L, H\}$, $x \geqslant 0$ and $\xi \in D(\widetilde{C})$, then*

$$e^{xC}\xi \in D(\widetilde{C})) \text{ and } \widetilde{C}e^{xC}\xi = e^{xC}\widetilde{C}\xi.$$

2. *Let $C \in \{M, L, L + M\}$, $\widetilde{C} \in \{M, L\}$, $x \geqslant 0$, $z \in X$ and $\lambda \in \rho\left(\widetilde{C}\right)$, then*

$$e^{xC}\left(\widetilde{C} - \lambda I\right)^{-1} z = \left(\widetilde{C} - \lambda I\right)^{-1} e^{xC} z.$$

3. *Let $C \in \{M, L\}$, then for $\xi \in D(\Lambda)$, $\lambda \in \rho(C)$ we have*

$$(C - \lambda I)^{-1}\xi \ \in D(\Lambda) \text{ and } (C - \lambda I)^{-1}\Lambda\xi = \Lambda(C - \lambda I)^{-1}\xi.$$

4. *Let $C \in \{M, L\}$ and $\xi \in D(C)$ we get $C\Lambda^{-1}\xi = \Lambda^{-1}C\xi$.*

5. *For $\xi \in D(\Lambda) = D(H) \cap D(L)$ we get $H\Lambda^{-1}\xi = \Lambda^{-1}H\xi$.*

6. *Let $x \geqslant 0$ then $L + M$ generates a uniformly bounded analytic semigroup in X satisfying $e^{x(L+M)} = e^{xL}e^{xM} = e^{xM}e^{xL}$.*

Proof.

1. Assume $x > 0$ and $\xi \in D(H)$. Operator C generates a C_0-semigroup, so we can apply the exponential formula (see [20, Theorem 8.3 p. 33]):
$$e^{xC}\xi = \lim_{n\to\infty}\left(\frac{n}{x}\left(\frac{n}{x}I - C\right)^{-1}\right)^n \xi, \tag{19}$$
and from (12), (15), (16) we deduce that
$$e^{xC}\widetilde{C}\xi = \lim_{n\to\infty}\left(\frac{n}{x}\left(\frac{n}{x}I - C\right)^{-1}\right)^n \widetilde{C}\xi = \lim_{n\to\infty}\widetilde{C}\left(\frac{n}{x}\left(\frac{n}{x}I - C\right)^{-1}\right)^n \xi,$$
then, since $\widetilde{C}$ is closed, we deduce that $e^{xC}\xi \in D(\widetilde{C})$ and $\widetilde{C}e^{xC}\xi = e^{xC}\widetilde{C}\xi$.

2. Setting $\xi = \left(\widetilde{C} - \lambda I\right)^{-1} z$, we deduce, from statement 1, that
$$\left(\widetilde{C} - \lambda I\right) e^{xC}\xi = e^{xC}\left(\widetilde{C} - \lambda I\right)\xi,$$
so $\left(\widetilde{C} - \lambda I\right) e^{xC}\left(\widetilde{C} - \lambda I\right)^{-1} z = e^{xC}z.$

3. Since $\xi \in D(H)$, from (15), (16) we deduce that $(C - \lambda I)^{-1}\xi \in D(\Lambda)$ and
$$\begin{aligned}\Lambda (C - \lambda I)^{-1}\xi &= \left((M - H) + e^{L+M}(L + H)\right)(C - \lambda I)^{-1}\xi \\ &= (M - H)(C - \lambda I)^{-1}\xi + e^{L+M}(L + H)(C - \lambda I)^{-1}\xi,\end{aligned}$$
now from (12), (15), (16) and statement 2, we deduce
$$\begin{aligned}\Lambda (C - \lambda I)^{-1}\xi &= (C - \lambda I)^{-1}(M - H)\xi + (C - \lambda I)^{-1}e^{L+M}(L + H)\xi \\ &= (C - \lambda I)^{-1}\Lambda\xi.\end{aligned}$$

4. We fix $\lambda \in \rho(C)$ and set $y = \Lambda^{-1}(C - \lambda I)\xi$, then from statement 3 we have
$$(C - \lambda I)^{-1}\Lambda y = \Lambda (C - \lambda I)^{-1} y,$$
that is $\xi = \Lambda (C - \lambda I)^{-1}\Lambda^{-1}(C - \lambda I)\xi$, so
$$(C - \lambda I)\Lambda^{-1}\xi = \Lambda^{-1}(C - \lambda I)\xi,$$
thus
$$C\Lambda^{-1}\xi = \Lambda^{-1}C\xi.$$

5. If $\xi \in D(\Lambda)$ we have $\Lambda\Lambda^{-1}\xi = \Lambda^{-1}\Lambda\xi$, that is
$$\left((M - H) + e^{L+M}(L + H)\right)\Lambda^{-1}\xi = \Lambda^{-1}\left((M - H) + e^{L+M}(L + H)\right)\xi,$$
so
$$\begin{aligned}&M\Lambda^{-1}\xi + e^{L+M}L\Lambda^{-1}\xi - \left(I - e^{L+M}\right)H\Lambda^{-1}\xi \\ &= \Lambda^{-1}M\xi + \Lambda^{-1}e^{L+M}L\xi - \Lambda^{-1}\left(I - e^{L+M}\right)H\xi,\end{aligned}$$
then
$$\left(I - e^{L+M}\right)H\Lambda^{-1}\xi = \Lambda^{-1}\left(I - e^{L+M}\right)H\xi = \left(I - e^{L+M}\right)\Lambda^{-1}H\xi.$$
But $I - e^{L+M}$ is boundedly invertible so $H\Lambda^{-1}\xi = \Lambda^{-1}H\xi$.

6. Applying statement 2, we get that, for $x \in (0,+\infty), n \in \mathbb{N}$

$$e^{xL}\left(\frac{n}{x}\left(\frac{n}{x}I-M\right)^{-1}\right)=\left(\frac{n}{x}\left(\frac{n}{x}I-M\right)^{-1}\right)e^{xL},$$

and by (19), we deduce that $e^{xL}e^{xM}=e^{xM}e^{xL}$. Then, $\left(e^{xL}e^{xM}\right)_{x\geqslant 0}$ is a strongly continuous semigroup (see [21, paragraph 5.15, p. 44]). We notice, that due to (14) $L+M$ is closed, then from paragraph 2.7, p. 64 in [21], we deduce that $L+M$ is the generator of the product semigroup $\left(e^{xL}e^{xM}\right)_{x\geqslant 0}$.

□

1.4. Representation of the Solution

We assume here (6) and (11)∼(17). Suppose that Problem (3) has a classical solution u. Then

$$u\in W^{2,p}(0,1;X)\cap L^p(0,1;D(LM)), u'\in L^p(0,1;D(L-M),$$

$u_0 := u(0) \in D(H)$ and one can write

$$\begin{cases} u''(.)+(L-M)u'(.)+LMu(.)\in L^p(0,1;X), \quad \text{a.e. } x\in(0,1)\\ u(0)=u_0, \quad u(1)=u_1. \end{cases}$$

Thus

$$u(0),u(1)\in\left(D\left(L^2\right),X\right)_{\frac{1}{2p},p}=\left(D\left(M^2\right),X\right)_{\frac{1}{2p},p}, \tag{20}$$

see ([4, Statement 2, Theorem 5, p. 173]). On the other hand, using (10), one obtains

$$\left(D\left(L^2\right),X\right)_{\frac{1}{2p},p}=(X,D(L))_{2-\frac{1}{p},q}=\left\{\phi\in D(L):L\phi\in(X,D(L))_{1-\frac{1}{p},q}\right\}.$$

Then

$$u(0),u(1)\in D(L)=D(M). \tag{21}$$

As in [3], u satifies, for a.e. $x\in(0,1)$

$$u(x)=e^{xM}\xi_0+e^{(1-x)L}\xi_1+I_x+J_x, \tag{22}$$

where

$$I_x=(L+M)^{-1}\int_0^x e^{(x-s)M}f(s)ds \text{ and } J_x=(L+M)^{-1}\int_x^1 e^{(s-x)L}f(s)ds.$$

Now, to obtain the final representation of u, it is enough to determine the constants ξ_0 and ξ_1 taking into account the boundary conditions of problem (3):

$$u'(0)-Hu(0)=d_0,\ u(1)=u_1. \tag{23}$$

It is clear that ξ_0, $\xi_1\in D(L)=D(M)$ due to (21), so

$$u(0)=\xi_0+e^L\xi_1+J_0.$$

We have, for a.e. $x\in(0,1)$

$$u'(x)=Me^{xM}\xi_0-Le^{(1-x)L}\xi_1+MI_x-LJ_x; \tag{24}$$

so

$$u'(0) = M\xi_0 - Le^L\xi_1 - LJ_0;$$

therefore

$$\begin{cases} \Lambda^{-1}u(0) = \Lambda^{-1}\xi_0 + \Lambda^{-1}e^L\xi_1 + \Lambda^{-1}J_0 \\ \Lambda^{-1}u'(0) = \Lambda^{-1}M\xi_0 - \Lambda^{-1}Le^L\xi_1 - \Lambda^{-1}LJ_0. \end{cases}$$

Since $\Lambda^{-1}(X) = D(H) \cap D(L) = D(H) \cap D(M)$ (due to (17)), we get

$$\begin{cases} H\Lambda^{-1}u(0) = H\Lambda^{-1}\xi_0 + H\Lambda^{-1}e^L\xi_1 + H\Lambda^{-1}J_0 \\ \Lambda^{-1}u'(0) = \Lambda^{-1}M\xi_0 - \Lambda^{-1}Le^L\xi_1 - \Lambda^{-1}LJ_0. \end{cases} \tag{25}$$

Then

$$\begin{aligned} \Lambda^{-1}d_0 &= \Lambda^{-1}\left[u'(0) - Hu(0)\right] = \Lambda^{-1}u'(0) - H\Lambda^{-1}u(0) \\ &= (M-H)\Lambda^{-1}\xi_0 - (L+H)\,\Lambda^{-1}e^L\xi_1 - (L+H)\,\Lambda^{-1}J_0, \end{aligned}$$

here we have used $H\Lambda^{-1}u(0) = \Lambda^{-1}Hu(0)$ (see Lemma 1, statement 5 and (21) and $M\Lambda^{-1} = \Lambda^{-1}M$ on $D(M), L\Lambda^{-1} = \Lambda^{-1}L$ on $D(L)$ (see Lemma 1, statement 4).

Now, from $u_1 = e^M\xi_0 + \xi_1 + I_1$, one obtains

$$\begin{aligned} \Lambda^{-1}d_0 &= \left[(M-H) + e^{L+M}(L+H)\right]\Lambda^{-1}\xi_0 - (L+H)\,\Lambda^{-1}e^L(u_1 - I_1) - (L+H)\,\Lambda^{-1}J_0 \\ &= \xi_0 - (L+H)\,\Lambda^{-1}e^L(u_1 - I_1) - (L+H)\,\Lambda^{-1}J_0, \end{aligned}$$

so

$$\xi_0 = \Lambda^{-1}d_0 + (L+H)\,\Lambda^{-1}\left[e^Lu_1 - e^LI_1 + J_0\right], \tag{26}$$

and

$$\xi_1 = -e^M(L+H)\,\Lambda^{-1}\left(e^Lu_1 - e^LI_1 + J_0\right) - e^M\Lambda^{-1}d_0 + u_1 - I_1. \tag{27}$$

Finally, we deduce the following representation of u

$$\begin{aligned} u(x) &= e^{xM}\left[\Lambda^{-1}d_0 + (L+H)\,\Lambda^{-1}e^Lu_1\right] - e^{xM}(L+H)\,\Lambda^{-1}e^L(L+M)^{-1}\int_0^1 e^{(1-s)M}f(s)ds \\ &+ e^{xM}(L+H)\,\Lambda^{-1}(L+M)^{-1}\int_0^1 e^{sL}f(s)ds \\ &+ e^{(1-x)L}\left[\left(I - (L+H)\Lambda^{-1}e^{L+M}\right)u_1 - \Lambda^{-1}e^Md_0\right] \\ &- e^{(1-x)L}(L+H)e^M\Lambda^{-1}(L+M)^{-1}\int_0^1 e^{sL}f(s)ds \\ &- e^{(1-x)L}\left[I - (L+H)\Lambda^{-1}e^{L+M}\right](L+M)^{-1}\int_0^1 e^{(1-s)M}f(s)ds \\ &+ (L+M)^{-1}\int_0^x e^{(x-s)M}f(s)ds + (L+M)^{-1}\int_x^1 e^{(s-x)L}f(s)ds, \end{aligned}$$

which can be written as

$$\begin{aligned} u(x) &= S(x, f_0, f, M) + S(1-x, f_1, f\,(1-.)\,, L) \\ &+ R(x, Tf_1, M) - R(1-x, f_0 + Te^L f_1, L), \end{aligned} \tag{28}$$

where

$$T = (L+H)\,\Lambda^{-1} \in \mathcal{L}(X), \tag{29}$$

$$f_0 = \Lambda^{-1} d_0 + T(L+M)^{-1} \int\limits_0^1 e^{sL} f(s) ds, \tag{30}$$

$$f_1 = u_1 - (L+M)^{-1} \int\limits_0^1 e^{(1-s)M} f(s) ds, \tag{31}$$

and for $\phi \in X$ and $C = L$ or M

$$\left\{ \begin{array}{l} S(x, \phi, f, C) = e^{xC}\phi + (L+M)^{-1} \int\limits_0^x e^{(x-s)C} f(s) ds \\ R(x, \phi, C) = e^{xC} e^{L+M-C}\phi. \end{array} \right. \tag{32}$$

This shows that if Problem (3) has a classical solution u then it is unique and determined by (28).

1.5. Technical Lemmas

We recall an important consequence of the Dore – Venni sum theory.

Lemma 2. *Assume (6) and let $-C \in$BIP(X, α) with $\alpha \in [0, \pi/2)$ and $1 < p < +\infty$. Then*

1. *For any $g \in L^p(0, 1; X)$*

$$x \longmapsto C \int_0^x e^{(x-s)C} g(s) ds \in L^p(0, 1; X). \tag{33}$$

2. *For any $g \in L^p(0, 1; X)$*

$$x \longmapsto C e^{xC} \int\limits_0^1 e^{sC} g(s) ds \in L^p(0, 1; X). \tag{34}$$

Proof. Statement 1 is proved in [13] when C is boundedly invertible and in [14] otherwise. Statement 2. is a consequence of statement 1 (see [3, p. 200, property (26)]).

□

Concerning interpolation recall that if C generates an analytic semigroup, then for all $q \in]1, +\infty]$, $m \in \mathbb{N} \setminus \{0\}$

$$\phi \in (D(C^m), X)_{1/mp,q} \Longleftrightarrow x \mapsto C^m e^{xC}\phi \in L^q(0, 1; X); \tag{35}$$

it follows, for example, that

$$\begin{cases} \phi \in (D(C), X)_{1/p,p} \Longleftrightarrow Ce^{\cdot C}\phi \in L^p(0,1;X) \\ \phi \in (D(C^2), X)_{1/2p,p} \Longleftrightarrow C^2e^{\cdot C}\phi \in L^p(0,1;X), \end{cases} \tag{36}$$

(see [16, p. 96]).

Lemma 3. *Assume (6) and (11)∼(14). Then, for $C = L$ or M and $\lambda_0 \in \rho(C)$ we have*

1. $\phi \in (D(C^2), X)_{1/2p,p} \Longleftrightarrow (C - \lambda_0 I)\, Ce^{\cdot C}\phi \in L^p(0,1;X)$.
2. $\phi \in (D(C^2), X)_{1/2p,p} \Longleftrightarrow (L + M - C)\, Ce^{\cdot C}\phi \in L^p(0,1;X)$.
3. $\phi \in (D(C^2), X)_{1/2p,p} \Longleftrightarrow (L + M - C)^2\, e^{\cdot C}\phi \in L^p(0,1;X)$.

Proof.

1. For the proof, see Lemma 3, p. 171, 172 in [4].

2. We assume that $C = L$ (the proof for $C = M$ is similar).
If $\phi \in (D(L^2), X)_{1/2p,p}$, then since $M(L-I)^{-1} \in \mathcal{L}(X)$, we obtain by statement 1

$$MLe^{\cdot L}\phi = M(L-I)^{-1}(L-I)\, Le^{\cdot L}\phi \in L^p(0,1;X).$$

Conversely, if $MLe^{\cdot L}\phi \in L^p(0,1;X)$ then

$$\begin{aligned} Le^{\cdot L}\phi &= L\,(M-I)\,(M-I)^{-1}\, e^{\cdot L}\phi \\ &= (M-I)^{-1}\, LMe^{\cdot L}\phi - L\,(M-I)^{-1}\, e^{\cdot L}\phi, \end{aligned}$$

so $Le^{\cdot L}\phi \in L^p(0,1;X)$, then from

$$\begin{aligned} L^2e^{\cdot L}\phi &= L^2\,(M-I)\,(M-I)^{-1}\, e^{\cdot L}\phi \\ &= L\,(M-I)^{-1}\, LMe^{\cdot L}\phi - L\,(M-I)^{-1}\, Le^{\cdot L}\phi, \end{aligned}$$

we deduce that $L^2e^{\cdot L}\phi \in L^p(0,1;X)$ and by (36) we get $\phi \in (D(L^2), X)_{1/2p,p}$.

3. We assume that $C = L$.
If $\phi \in (D(L^2), X)_{1/2p,p}$, then $Me^{\cdot L}\phi \in L^p(0,1;X)$ since

$$\begin{aligned} Me^{\cdot L}\phi &= M(L-I)^{-1}\,(L-I)\, e^{\cdot L}\phi \\ &= M(L-I)^{-1}Le^{\cdot L}\phi - M(L-I)^{-1}e^{\cdot L}\phi, \end{aligned}$$

by statement 2, we deduce that $M^2e^{\cdot L}\phi \in L^p(0,1;X)$ since

$$\begin{aligned} M^2e^{\cdot L}\phi &= M(L-I)^{-1}M\,(L-I)\, e^{\cdot L}\phi \\ &= M(L-I)^{-1}MLe^{\cdot L}\phi - M(L-I)^{-1}Me^{\cdot L}\phi. \end{aligned}$$

Conversely, if $M^2 e^{\cdot L}\phi \in L^p(0,1;X)$ then $Me^{\cdot L}\phi \in L^p(0,1;X)$ since

$$\begin{aligned} Me^{\cdot L}\phi &= M(M-I)(M-I)^{-1} e^{\cdot L}\phi \\ &= (M-I)^{-1} M^2 e^{\cdot L}\phi - M(M-I)^{-1} e^{\cdot L}\phi, \end{aligned}$$

but

$$\begin{aligned} LMe^{\cdot L}\phi &= LM(M-I)(M-I)^{-1} e^{\cdot L}\phi \\ &= L(M-I)^{-1} M^2 e^{\cdot L}\phi - L(M-I)^{-1} Me^{\cdot L}\phi, \end{aligned}$$

so $LMe^{\cdot L}\phi \in L^p(0,1;X)$, and statement 2 gives $\phi \in (D(L^2),X)_{1/2p,p}$.

□

We are now in position to study the regularity of the terms R, S appearing in (28).

Lemma 4. *Assume (6), (11)∼(17). Let $C = L$ or M and ϕ a given element in X. Then for the regular term defined above $R(.,\phi,C)$ we have*

$$LMR(\cdot,\phi,C),\ L^2R(\cdot,\phi,C),\ M^2R(\cdot,\phi,C) \in L^p(0,1;X).$$

For the proof of this lemma see Lemma 2, p. 170, 171 in [4].

Now concerning the singular term $S(\cdot,f_0,f,M)$, we have:

Proposition 1. *Assume (6) and (11)∼(17). Let $f \in L^p(0,1;X)$, $1 < p < +\infty$ and $P \in \{LM, M^2, L^2\}$ then*

$$PS(\cdot,f_0,f,M) \in L^p(0,1;X) \Longleftrightarrow \Lambda^{-1}d_0 \in (D(LM),X)_{\frac{1}{2p},p},$$

(here f_0 is given by (30)).

Proof. We set, for a.e. $x \in (0,1), C \in \{L,M\}$

$$\begin{cases} \mathcal{L}(x,g,C) = LM(L+M)^{-1} \int_0^x e^{(x-s)C} f(s)ds \\ \mathcal{M}(x,g,C) = LM(L+M)^{-1} e^{xC} \int_0^1 e^{s(L+M-C)} g(s)ds, \end{cases} \tag{37}$$

in virtue of the commutativity of L, M, one can write for $y \in D(L) = D(M)$

$$LM(L+M)^{-1}y = (L+M-C)(L+M)^{-1}Cy,$$

from which we get

$$\begin{cases} \mathcal{L}(x,g,C) = (L+M-C)(L+M)^{-1}C \int_0^x e^{(x-s)C} f(s)ds \\ \mathcal{M}(x,g,C) = (L+M-C)(L+M)^{-1}Ce^{xC} \int_0^1 e^{s(L+M-C)} g(s)ds. \end{cases}$$

Since $(L+M-C)(L+M)^{-1} \in \mathcal{L}(X)$, one has

$$\mathcal{L}(\cdot,g,C) \in L^p(0,1;X), \tag{38}$$

and

$$x \longmapsto (L+M-C)\, e^{x(L+M-C)} \int_0^1 e^{s(L+M-C)} g(s)ds \in L^p(0,1;X),$$

thus

$$\int_0^1 e^{s(L+M-C)} g(s)ds \in \left(D\left(L+M-C\right), X\right)_{\frac{1}{p},p} = \left(D\left(C\right), X\right)_{\frac{1}{p},p},$$

see (36) and again

$$x \longmapsto Ce^{xC} \int_0^1 e^{s(L+M-C)} g(s)ds \in L^p(0,1;X),$$

from which it follows that

$$\mathcal{M}(\cdot, g, C) \in L^p(0,1;X). \tag{39}$$

For example if $P = LM$ then we write, for a.e. $x \in (0,1)$

$$\begin{aligned}
PS(x, f_0, f, M) &= LMe^{xM} f_0 + LM(L+M)^{-1} \int_0^x e^{(x-s)M} f(s)ds \\
&= LMe^{xM}\Lambda^{-1} d_0 + LM(L+M)^{-1} \int_0^x e^{(x-s)M} f(s)ds \\
&+ LMe^{xM}\left(L+H\right)\Lambda^{-1}(L+M)^{-1} \int_0^1 e^{sL} f(s)ds \\
&= Me^{xM} L\Lambda^{-1} d_0 + \mathcal{L}(x, f, M) + \left(L+H\right)\Lambda^{-1}\mathcal{M}(x, f, M),
\end{aligned}$$

and since $L\Lambda^{-1}, H\Lambda^{-1} \in \mathcal{L}\left(X\right)$, from (38), (39) and (36) we deduce that

$$\begin{aligned}
PS(\cdot, f_0, f, M) \in L^p(0,1;X) &\iff Me^{\cdot M} L\Lambda^{-1} d_0 \in L^p(0,1;X) \\
&\iff L\Lambda^{-1} d_0 \in (D(M), X)_{\frac{1}{p},p} = (D(L), X)_{\frac{1}{p},p} \\
&\iff \Lambda^{-1} d_0 \in (X, D(L))_{2-\frac{1}{p},p} \\
&\iff \Lambda^{-1} d_0 \in (X, D(L^2))_{1-\frac{1}{2p},p} \\
&\iff \Lambda^{-1} d_0 \in (D(L^2), X)_{\frac{1}{2p},p} \\
&\iff \Lambda^{-1} d_0 \in (D(LM), X)_{\frac{1}{2p},p}.
\end{aligned}$$

The cases $P = L^2$ or M^2 are similarly treated.

□

For the term $S(1-.,f_1, f\left(1-.\right), L)$ we have

Proposition 2. *Assume (6) and (11)∼(17). Let* $f \in L^p(0,1;X)$ *and* $P \in \{LM, M^2, L^2\}$ *then*

$$PS(1-\cdot, f_1, f\left(1-.\right), L) \in L^p(0,1;X) \iff u_1 \in (D(LM), X)_{\frac{1}{2p},p},$$

(here f_1 *is given by (31)).*

Proof. Assume $P = LM$ (the cases $P = M^2$ and $P = L^2$ are similarly treated), one has

$$\begin{aligned} & PS(1-x, f_1, f(1-.), L) \\ = \; & LMe^{(1-x)L} f_1 + LM(L+M)^{-1} \int_0^{1-x} e^{(1-x-s)L} f(1-s) ds \\ = \; & LMe^{(1-x)L} u_1 + LM(L+M)^{-1} \int_0^{1-x} e^{(1-x-s)L} f(1-s) ds \\ - \; & LM(L+M)^{-1} e^{(1-x)L} \int_0^1 e^{sM} f(1-s) ds \\ = \; & LMe^{(1-x)L} u_1 + \mathcal{L}(1-x, f(1-\cdot), L) - \mathcal{M}(1-x., f(1-\cdot), L), \end{aligned}$$

then by Lemma 3, statement 2

$$\begin{aligned} PS(1-., f_1, f(1-.), L) \in L^p(0,1;X) & \iff LMe^{(1-\cdot)L} u_1 \in L^p(0,1;X) \\ & \iff u_1 \in (D(L^2), X)_{\frac{1}{2p},p} . \end{aligned}$$

□

1.6. Main Result for Problem (3)

Theorem 1. *Assume (6) and (11)∼(17). Let $f \in L^p(0,1;X)$ with $1 < p < \infty$. Then Problem (3) has a classical solution u if and only if*

$$\Lambda^{-1} d_0, u_1 \in (D(LM), X)_{1/2p,p} .$$

In this case, u is uniquely determined by (28).

Proof. From subsection 1.4. we know that if (3) has a classical solution u then

$$\begin{aligned} u(x) & = S(x, f_0, f, M) + S(1-x, f_1, f(1-.), L) \\ & + R(x, Tf_1, M) - R(1-x, f_0 + Te^L f_1, L), \end{aligned} \tag{40}$$

where f_0, f_1 and T are given in (30), (31) and (29). To conclude it is enough to study the regularity of (40). From Lemma 4, one has

$$LMR(x, Tf_1, M) - LMR(1-x, f_0 + Te^L f_1, L) \in L^p(0,1;X),$$

and Lemmas 1 and 2 give

$$\begin{cases} LMS(., f_0, f, M) \in L^p(0,1;X) \iff \Lambda^{-1} d_0 \in (D(LM, X)_{\frac{1}{2p},p} \\ LMS(1-., f_1, f(1-.), L) \in L^p(0,1;X) \iff u_1 \in (D(LM), X)_{1/2p,p} . \end{cases}$$

Summarizing, we obtain

$$LMu \in L^p(0,1;X) \iff \Lambda^{-1} d_0, u_1 \in (D(LM), X)_{1/2p,p} .$$

On the other hand

$$\begin{aligned}(L-M)\,u'(x) &= (L-M)\,MS(x,f_0,f,M)+(L-M)\,LR(1-x,f_0+Te^{L}f_1,L)\\ &- (L-M)\,LS(1-x,f_1,f\,(1-.)\,,L)+(L-M)\,MR(x,Tf_1,M)\\ &= \left(LM-M^2\right)S(x,f_0,f,M)+\left(L^2-LM\right)R(1-x,f_0+Te^{L}f_1,L)\\ &- \left(L^2-LM\right)S(1-x,f_1,f\,(1-.)\,,L)+\left(LM-M^2\right)R(x,Tf_1,M).\end{aligned}$$

Using again Lemma 4 and Propositions 1 and 2 we obtain

$$(L-M)u'(.)\in L^p(0,1;X)\Longleftrightarrow \Lambda^{-1}d_0,u_1\in(D(LM),X)_{1/2p,p}\,.$$

So u have the desired regularities.

Now we will conclude by showing that the fonction u given by (28), satisfies (3). One has

$$\begin{aligned}u''(x) &= M^2S(x,f_0,f,M_\omega)-L^2R(1-x,f_0+Te^{L}f_1,L)\\ &+ L^2S(1-x,f_1,f\,(1-.)\,,L)+M^2R(x,Tf_1,M)+f(x),\end{aligned}\tag{41}$$

using the fact that $M^2+(L-M)M-LM=L^2-(L-M)L-LM\subset 0$ and (28), (41) and (41), we obtain

$$\begin{aligned}&u''(x)+(L-M)u'(x)-LMu(x)=\left[M^2+(L-M)M-LM\right]S(x,f_0,f,M)\\ &-\left[L^2-(L-M)L-LM\right]R(1-x,f_0+Te^{L}f_1,L)\\ &+\left[L^2-(L-M)L-LM\right]S(1-x,f_1,f\,(1-.)\,,L)\\ &+\left[M^2+(L-M)M-LM\right]R(x,Tf_1,M)+f(x)=f(x).\end{aligned}$$

From (28), we have

$$u(1)=S(1,f_0,f,M)-R(0,f_0+Te^{L}f_1,L)+S(0,f_1,f\,(1-.)\,,L)+R(1,Tf_1,M),$$

so

$$\begin{aligned}&u(1)=e^{M}f_0+(L+M)^{-1}\int_0^1 e^{(1-s)M}f(s)ds-e^{M}\left[f_0+(L+H)\,\Lambda^{-1}e^{L}f_1\right]\\ &+f_1+(L+H)\,\Lambda^{-1}e^{L+M}f_1=f_1+(L+M)^{-1}\int_0^1 e^{(1-s)M}f(s)ds=u_1,\end{aligned}$$

and

$$\begin{aligned}&u(0)=S(0,f_0,f,M)+S(1,f_1,f\,(1-.)\,,L)+R(0,Tf_1,M)-R(1,f_0+Te^{L}f_1,L)\\ &=f_0+e^{L}f_1+(L+M)^{-1}\int_0^1 e^{sL}f(s)ds+(L+H)\,\Lambda^{-1}e^{L}f_1-e^{L+M}\left(f_0+(L+H)\,\Lambda^{-1}e^{L}f_1\right)\\ &=\left(I-e^{L+M}\right)f_0+\left[I+\left(I-e^{L+M}\right)(L+H)\,\Lambda^{-1}\right]e^{L}f_1+(L+M)^{-1}\int_0^1 e^{sL}f(s)ds,\end{aligned}$$

so

$$u(0) = \Lambda^{-1}\left(I - e^{L+M}\right) d_0 + \Lambda^{-1}(L+M)\, e^L \left[u_1 - (L+M)^{-1} \int_0^1 e^{(1-s)M} f(s)ds \right]$$

$$+ \left[\Lambda + \left(I - e^{L+M}\right)(L+H)\right] \Lambda^{-1}(L+M)^{-1} \int_0^1 e^{sL} f(s)ds.$$

Moreover, we have $\Lambda + \left(I - e^{L+M}\right)(L+H) \subset L+M$ and

$$\left(I - e^{L+M}\right)(L+H)\,\Lambda^{-1} + I = (L+M)\,\Lambda^{-1},$$

then

$$\begin{aligned} u(0) &= \Lambda^{-1}\left(I - e^{L+M}\right) d_0 \\ &+ \Lambda^{-1}(L+M)\, e^L \left[u_1 - (L+M)^{-1} \int_0^1 e^{(1-s)M} f(s)ds \right] + \Lambda^{-1} \int_0^1 e^{sL} f(s)ds, \end{aligned}$$

from which we notice a good surprise: $u(0) \in D(H)$ and

$$u'(0) = MS(0, f_0, f, M) + LR(1, f_0 + Te^L f_1, L) - LS(1, f_1, f\,(1-.)\,, L) + MR(0, Tf_1, M).$$

Therefore

$$\begin{aligned} &u'(0) - Hu(0) = (M-H)\, S(0, f_0, f, M) + (L+H)\, R(1, f_0 + Te^L f_1, L) \\ &- (L+H)\, S(1, f_1, f\,(1-.)\,, L) + (M-H)\, R(0, Tf_1, M) \\ &= (M-H)\, f_0 + (L+H)\, e^{L+M}\left(f_0 + (L+H)\,\Lambda^{-1} e^L f_1\right) - (L+H)\, e^L f_1 \\ &- (L+H)\,(L+M)^{-1} \int_0^1 e^{(1-s)L} f(1-s)ds + (M-H)\,(L+H)\,\Lambda^{-1} e^L f_1 \\ &= \Lambda f_0 + (L+H)\left[(M-H) + e^{L+M}(L+H) - \Lambda\right] \Lambda^{-1} e^L f_1 \\ &- (L+H)\,(L+M)^{-1} \int_0^1 e^{(1-s)L} f(1-s)ds \\ &= d_0 + (L+H)\,(L+M)^{-1} \int_0^1 e^{sL} f(s)ds - (L+H)\,(L+M)^{-1} \int_0^1 e^{sL} f(s)ds = d_0. \end{aligned}$$

□

Remark 4. Assume (6) and (11)~(17). If

$$d_0 \in (D(M), X)_{1/p,p}\,, u_1 \in \left(D(M^2), X\right)_{1/2p,p}\,,$$

then

$$\Lambda^{-1} d_0, u_1 \in \left(D(M^2), X\right)_{1/2p,p}\,,$$

since $\Lambda^{-1}(X) \subset D(L) = D(M)$. So Problem (3) admits a classical solution u.

2. Go Back to Problem (1)

2.1. Hypotheses on Operators A, B and H

Our essential assumption on operators A, B (which means the ellipticity of our equation) is the following

$$\begin{cases} B^2 - A \text{ is a linear closed operator in } X, \ \ ]-\infty, 0] \subset \rho(B^2 - A) \text{ and} \\ \sup\limits_{\lambda \geqslant 0} \left\| \lambda \left(\lambda I + B^2 - A \right)^{-1} \right\|_{\mathcal{L}(X)} < +\infty, \end{cases} \tag{42}$$

(it follows that the operator $-(B^2 - A)^{1/2}$ is the infinitesimal generator of an analytic semigroup on X, see for instance [22]).

$$D((B^2 - A)^{1/2}) \subset D(B), \tag{43}$$

Now, set

$$L = B - (B^2 - A)^{1/2} \quad \text{and} \quad M = -B - (B^2 - A)^{1/2},$$

then, we will assume that

$$\exists \theta_L, \theta_M \in]0, \pi/2[: -L \in \mathrm{BIP}\left(X, \theta_L\right), \ -M \in \mathrm{BIP}\left(X, \theta_M\right), \tag{44}$$

$$\forall y \in D(B), \ \left(B^2 - A\right)^{-1/2} By = B \left(B^2 - A\right)^{-1/2} y, \tag{45}$$

$$\begin{cases} \forall \xi \in D(H), \forall \lambda \in \rho(L), \ (L - \lambda I)^{-1} \xi \in D(H) \text{ and} \\ (L - \lambda I)^{-1} H\xi = H \left(L - \lambda I\right)^{-1} \xi, \end{cases} \tag{46}$$

$$\begin{cases} \forall \xi \in D(H), \forall \mu \in \rho(M), \ (M - \mu I)^{-1} \xi \in D(H) \text{ and} \\ (M - \mu I)^{-1} H\xi = H \left(M - \mu I\right)^{-1} \xi. \end{cases} \tag{47}$$

In addition, setting $D(\Lambda) = D(L) \cap D(H)$ and $\Lambda = (M - H) + e^{L+M}\left(L + H\right)$, we will also suppose that

$$\Lambda \text{ is closed and boundedly invertible.} \tag{48}$$

Remark 5. Assume (42)$\sim$(46). Then

1. $D(L) = D(M) = D((B^2 - A)^{1/2})$ so

$$D(L - M) = D(L + M) = D((B^2 - A)^{1/2}) \subset D\left(B\right),$$

   thus $L - M \subset 2B, L + M = -2(B^2 - A)^{1/2}$ and $0 \in \rho\left(L + M\right)$.

2. $D(ML) = D(LM) = D(B^2 - A)$ and $ML = LM \subset -A$.

For the proof, see Lemma 7 in [4].

2.2. Main Result for Problem (1)

Theorem 2. *Assume (6) and (42)∼(48). Let $f \in L^p(0,1;X)$ with $1 < p < \infty$. Then Problem (1) has a classical solution u satisfying moreover*

$$u \in L^p(0,1;D(B^2-A)) \text{ and } u' \in L^p(0,1;D((B^2-A)^{1/2}),$$

if and only if

$$\Lambda^{-1}d_0 \in \left(D(B^2-A)), X\right)_{1/2p,p} \text{ and } u_1 \in \left(D(B^2-A), X\right)_{1/2p,p}.$$

In this case, u is uniquely determined by (28).

Proof. If we assume (42)∼(48), then hypotheses (11)∼(17) are satisfied with L, M, Λ defined as in subsection 2.1. Thus we can apply Theorem 1.

□

3. Some Cases in Which Assumption (17) or (48) are Satisfied

3.1. New Assumptions Implying (17)

Proposition 3. *Assume (6) and (11)∼(16). If*

$$M - H \text{ is closed and } 0 \in \rho(M-H), \tag{49}$$

and

$$\left\| \left(I - e^{L+M}\right)^{-1} (L+M) e^{L+M} (M-H)^{-1} \right\|_{\mathcal{L}(X)} < 1, \tag{50}$$

then assumption (17) is satisfied and we can apply Theorem 1.

Proof. Since $I - e^{L+M}$ is boundedly invertible (see [19, p. 60]), we can write

$$\begin{aligned} \Lambda &= (M-H) - e^{L+M}\left[(M-H) - L - M\right] \\ &= \left(I - e^{L+M}\right)\left[I + \left(I - e^{L+M}\right)^{-1}(L+M)e^{L+M}(M-H)^{-1}\right](M-H) = G(M-H), \end{aligned}$$

where $G := \left(I - e^{L+M}\right)\left[I + \left(I - e^{L+M}\right)^{-1}(L+M)e^{L+M}(M-H)^{-1}\right] \in \mathcal{L}(X)$.

Now, $0 \in \rho(G)$ due to (50). Using (49) we deduce that $\Lambda = G(M-H)$ is boundedly invertible.

□

Remark 6.

1. Proposition 3 remains true if we replace (50) by

$$\begin{cases} \text{for some } n_1 \in \mathbb{N}\backslash\{0\} \\ \left\| \left[\left(I - e^{L+M}\right)^{-1}(L+M)e^{L+M}(M-H)^{-1}\right]^{n_1} \right\|_{\mathcal{L}(X)} < 1. \end{cases}$$

2. Assumption (49) can be obtained, for instance, in the following manner : under (6) and (11)∼(16), if we assume in addition that

$$\begin{cases} -M \in \mathrm{BIP}\,(X, \theta_M) \\ H \in \mathrm{BIP}\,(X, \theta_H)\,, \text{ with } \theta_H \in (0, \pi) \\ 0 \in \rho\,(M) \cup \rho\,(H) \ \text{ and } \theta_M + \theta_H \in (0, \pi), \end{cases}$$

then $(-M) + H$ is closed and boundedly invertible (see [14, Theorem 4, p. 441 together with the remark at the end of p. 445]), that is (49).

3. The spectral Problem with a parameter $\omega \geqslant \omega_0$ (where $\omega_0 \geqslant 0$ is some fixed number)

$$\begin{cases} u''(x) + 2Bu'(x) + Au(x) - \omega u(x) = f(x), \quad \text{a.e. } x \in (0,1), \\ u'(0) - Hu(0) = d_0,\ u(1) = u_1, \end{cases} \tag{51}$$

is studied in [24], as an application of this paper, setting

$$A_\omega = A - \omega I, L_\omega = B - \left(B^2 - A_\omega\right)^{1/2} \text{ and } M_\omega = -B - \left(B^2 - A_\omega\right)^{1/2},$$

Problem (51) becomes

$$\begin{cases} u''(x) + (L_\omega - M_\omega)\, u'(x) - L_\omega M_\omega u(x) = f(x), \quad \text{a.e. } x \in (0,1), \\ u'(0) - Hu(0) = d_0,\ u(1) = u_1, \end{cases}$$

and, under suitable assumptions, we can apply the results of this paper, replacing L, M by L_ω, M_ω. In particular, the spectral parameter ω is used to obtain (50) for ω large enough and then (17) by Proposition 3.

Proposition 4. *Assume (6) and (11)∼(16). If*

$$M - H \textit{ is closed and } 0 \in \rho\,(M - H)\,,$$

and, for some $n_1 \in \mathbb{N}\backslash\{0\}$

$$\left\| \left((L + H)\,(M - H)^{-1} \right)^{n_1} \right\|_{\mathcal{L}(X)} \leqslant 1,$$

then assumption (17) is satisfied and we can apply Theorem 1.

Proof. We write

$$\Lambda = \left[I + e^{L+M}\,(L + H)\,(M - H)^{-1} \right](M - H) = (I - C)\,(M - H)\,,$$

where $C = -e^{L+M}\,(L + H)\,(M - H)^{-1}$. So (17) will be satisfied if and only if $0 \in \rho(I - C)$. We proceed as in the proof of Lemma 2.3, p. 1458 in [10].

- $\exists K \geqslant 1, \exists\, \delta > 0, \forall y > 0 : \|e^{y(L+M)}\|_{\mathcal{L}(X)} \leqslant Ke^{-\delta y}$.
- $\exists k \in \mathbb{N}\backslash\{0\} : \left\|e^{2kn_1(L+M)}\right\|_{\mathcal{L}(X)} \leqslant Ke^{-2kn_1\delta} < 1$.

So $\left\|C^{kn_1}\right\|_{\mathcal{L}(X)} < 1$ then $0 \in \rho(I - C^{kn_1})$ and thus $0 \in \rho(I - C)$. □

3.2. Some Particular Cases

Consider Problem 3 and assume (6) and (11)∼(16).

1. If $H = -L$ then $\Lambda = M + L$, and (17) is satisfied.
2. If $-\dfrac{1}{2}(L - M) \subset H$ then $\Lambda = \dfrac{1}{2}(L + M)\left(I + e^{L+M}\right)$ and again (17) is satisfied.

Similarly, consider Problem 1 and assume (42)∼(47). If $H = -B + \sqrt{B^2 - A}$ or $-B$ then (48) is satisfied.

4. Applications

Example 1. Consider K such that $-K$ has bounded imaginary powers and $0 \in \rho(K)$. Take

$$L = M = -H = -\sqrt{-K}.$$

Then $\Lambda = M + L = -2\sqrt{-K}$ is boundedly invertible and

$$-L = (-K)^{1/2},\ (-L)^{it} = (-K)^{it/2}\ \ (t \in \mathbb{R}).$$

Moreover $L - M \subset 0$ and $-ML = K$, so we can apply our main result to the following Problem

$$\begin{cases} u''(x) + Ku(x) = f(x), & \text{a.e. } x \in (0,1) \\ u'(0) - \sqrt{-K}u(0) = d_0,\ u(1) = u_1. \end{cases}$$

For example, take $X = L^p(\Omega)$, with $1 < p < \infty$, Ω a bounded domain in $\mathbb{R}^n$ with smooth boundary, and, $K = \Delta - cI$, $(c > 0)$ with Dirichlet boundary conditions or the conditions described in [16, p. 320]. Then the fractional power $\sqrt{-\Delta + cI}$ is well defined.

Example 2. Consider $X = L^2(\mathbb{R})$ and L, M operators in X defined by

$$\begin{cases} D(L) = D(M) = H^2(\mathbb{R}) \\ Lu = a\dfrac{\partial^2 u}{\partial y^2} + b\dfrac{\partial u}{\partial y} + cu,\ Mu = a\dfrac{\partial^2 u}{\partial y^2} \\ (a, b, c \in \mathbb{R}, a > 0, c < 0), \end{cases}$$

so that $(L - M)u = b\dfrac{\partial u}{\partial y} + cu$ and $(L + M)u = 2a\dfrac{\partial^2 u}{\partial y^2} + b\dfrac{\partial u}{\partial y} + cu$ with

$$0 \in \rho(L + M).$$

Take $Hu = -\dfrac{1}{2}\left(b\dfrac{\partial u}{\partial y} + cu\right), u \in D(H) := H^1(\mathbb{R})$.

Then our main result applies to

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \left(b\dfrac{\partial^2 u}{\partial y \partial x} + c\dfrac{\partial u}{\partial x}\right)(x,y) - a\left(a\dfrac{\partial^4 u}{\partial y^4} + b\dfrac{\partial^3 u}{\partial y^3} + c\dfrac{\partial^2 u}{\partial y^2}\right)(x,y) \\ = f(x,y), \quad x \in (0,1), y \in \mathbb{R} \\ \dfrac{\partial u}{\partial x}(0,y) + \dfrac{b}{2}\dfrac{\partial u}{\partial y}(0,y) + \dfrac{c}{2}u(0,y) = d_0(y), \quad y \in \mathbb{R} \\ u(1,y) = u_1(y), \quad y \in \mathbb{R}. \end{cases}$$

Of course, this example can be generalized to $\mathbb{R}^n$ and differential operators

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right)u + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu,$$

(see [14, 23]).

Example 3. Choose L, M satisfying (11)∼(15) and H satisfying (16), assume moreover that $M - H$ has a bounded inverse. It remains to obtain (17). But, as indicated in Proposition 4, we have

$$\Lambda = \left[I + e^{L+M}(L+H)(M-H)^{-1}\right](M-H),$$

with $(L+H)(M-H)^{-1} \in \mathcal{L}(X)$ and the invertibility of Λ is guaranted by the smallness of

$$\left\|e^{L+M}(L+H)(M-H)^{-1}\right\|_{\mathcal{L}(X)}.$$

Now, if we replace L by $L_\delta := L - \delta I$ with $\delta > 0$ large enough, then L_δ, M and H satisfy (11)∼(17), indeed:

$$\begin{gathered}\left\|e^{L_\delta+M}(L_\delta+H)(M-H)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \left\|e^{L_\delta}\right\|_{\mathcal{L}(X)}\left\|e^{M}(L+H)(M-H)^{-1}\right\|_{\mathcal{L}(X)} \\ +\delta\left\|e^{L_\delta}\right\|_{\mathcal{L}(X)}\left\|e^{M}(M-H)^{-1}\right\|_{\mathcal{L}(X)} \leq e^{-\delta}\left\|e^{L}\right\|_{\mathcal{L}(X)}\left\|e^{M}(L+H)(M-H)^{-1}\right\|_{\mathcal{L}(X)} \\ +\delta e^{-\delta}\left\|e^{L}\right\|_{\mathcal{L}(X)}\left\|e^{M}(M-H)^{-1}\right\|_{\mathcal{L}(X)},\end{gathered}$$

and then $\left\|e^{L_\delta+M}(L_\delta+H)(M-H)^{-1}\right\|_{\mathcal{L}(X)} < 1$ for $\delta > 0$ large enough.

This immediately apply to the differential operators handled in the papers of Prüss and Sohr quoted above.

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ЭЛЛИПТИЧЕСКИЕ ЗАДАЧИ С ГРАНИЧНЫМИ ОПЕРАТОРНО-КОЭФФИЦИЕНТНЫМИ УСЛОВИЯМИ РОБИНА В L_p ПРОСТРАНСТВАХ СОБОЛЕВА И ИХ ПРИЛОЖЕНИЯ

М. Чеггаг, А. Фавини, Р. Лаббас, С. Менго, А. Медегри

В статье доказаны некоторые новые результаты о полных операторно-дифференциальных уравнениях эллиптического типа второго порядка с граничными операторно-коэффициентными условиями Робина в пространстве $L^p(0,1;X)$ в случае, когда $p \in (1,+\infty)$, а X — банахово UMD-пространство. Доказано существование, единственность и оптимальная регулярность классического решения. Статья дополняет и завершает предыдущие исследования авторов по данной проблематике.

Ключевые слова: абстрактные эллиптические дифференциальные уравнения второго порядка; граничные условия Робина; аналитическая полугруппа.

Мустафа Чеггаг, Национальная политехническая школа (г. Оран, Алжир), cheggag_m@yahoo.fr.

Анджело Фавини, кафедра математики, Болонский университет (г. Болонья, Италия), favini@dm.unibo.it.

Раба Лаббас, лаборатория математики, Университет Гавра (г. Гавр, Франция), rabah.labbas@univ-lehavre.fr.

Стефан Менго, лаборатория математики, Университет Гавра (г. Гавр, Франция), stephane.maingot@univ-lehavre.fr.

Ахмед Медегри, лаборатория теоретической и прикладной математики, Университет Мостаганем (Алжир), medeghri@univ-mosta.dz.

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